

ON MAPS BETWEEN MODULAR JACOBIANS AND JACOBIANS OF SHIMURA CURVES

BY

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ABSTRACT

Fix a squarefree integer N , divisible by an even number of primes, and let Γ' be a congruence subgroup of level M , where M is prime to N . For each D dividing N and divisible by an even number of primes, the Shimura curve $X^D(\Gamma_0(N/D) \cap \Gamma')$ associated to the indefinite quaternion algebra of discriminant D and $\Gamma_0(N/D) \cap \Gamma'$ -level structure is well defined, and we can consider its Jacobian $J^D(\Gamma_0(N/D) \cap \Gamma')$. Let J^D denote the N/D -new subvariety of this Jacobian.

By the Jacquet–Langlands correspondence [J-L] and Faltings’ isogeny theorem [Fa], there are Hecke-equivariant isogenies among the various varieties J^D defined above. However, since the isomorphism of Jacquet–Langlands is noncanonical, this perspective gives no information about the isogenies so obtained beyond their existence. In this paper, we study maps between the varieties J^D in terms of the maps they induce on the character groups of the tori corresponding to the mod p reductions of these varieties for p dividing N . Our characterization of such maps in these terms allows us to classify the possible kernels of maps from J^D to $J^{D'}$, for D dividing D' , up to support on a small finite set of maximal ideals of the Hecke algebra. This allows us to compute the Tate modules $T_{\mathfrak{m}} J^D$ of J^D at all non-Eisenstein \mathfrak{m} of residue characteristic $l > 3$. These computations have implications for the multiplicities of irreducible Galois representations in the torsion of Jacobians of Shimura curves; one such consequence is a “multiplicity one” result for Jacobians of Shimura curves.

1. Introduction

The Jacquet–Langlands correspondence asserts the existence of noncanonical isomorphisms between spaces of modular forms coming from different quaternion algebras over \mathbb{Q} . More precisely, if D is a squarefree product of an even number of primes, p and q are distinct primes not dividing D , and B and B' are quaternion algebras over \mathbb{Q} with discriminants D and Dpq , respectively, then the Jacquet–Langlands correspondence asserts the existence of an isomorphism:

$$S_2^{Dpq}(\Gamma) \cong S_2^D(\Gamma_0(pq) \cap \Gamma)_{pq\text{-new}}.$$

Here $S_2^{Dpq}(\Gamma)$ denotes the space of weight two modular forms associated to the quaternion algebra B' and the congruence subgroup Γ (which must be of level coprime to Dpq). Similarly, $S_2^D(\Gamma_0(pq) \cap \Gamma)$ is the space of weight two modular forms associated to the quaternion algebra B and the congruence subgroup $\Gamma_0(pq) \cap \Gamma$. The subscript pq -new denotes the subspace of $S_2^D(\Gamma_0(pq) \cap \Gamma)$ consisting of forms that are “new” for both p and q .

The existence of such an isomorphism has consequences for the geometry of modular curves and Shimura curves. In particular, let $X^{Dpq}(\Gamma)$ denote the Shimura curve associated to the quaternion algebra B' and the congruence subgroup Γ , and define $X^D(\Gamma_0(pq) \cap \Gamma)$ similarly. Then we may consider their Jacobians $J^{Dpq}(\Gamma)$ and $J^D(\Gamma_0(pq) \cap \Gamma)$. The Jacquet–Langlands correspondence, together with Eichler–Shimura theory, tells us that, after tensoring with \mathbb{Q} , the l -adic Tate module of $J^{Dpq}(\Gamma)$ is $G_{\mathbb{Q}}$ -equivariantly isomorphic to the corresponding Tate module of the pq -new subvariety of $J^D(\Gamma_0(pq) \cap \Gamma)$. It follows from [Ri3] or from Faltings’ isogeny theorem [Fa], that J^{Dpq} is isogenous to the pq -new subvariety of $J^D(\Gamma_0(pq) \cap \Gamma)$.

Actually producing an isogeny between these two abelian varieties, on the other hand, seems much more difficult. The above argument, while enough to establish the existence of such an isogeny, provides no hint as to how an explicit map might be constructed.

The first results to give geometric information about the relationship between $J^{Dpq}(\Gamma)$ and $J^D(\Gamma_0(pq) \cap \Gamma)$ are due to Ribet [Ri1], who established an isomorphism between the character group of the torus associated to the mod p reduction of $J^{Dpq}(\Gamma)$ and a certain subgroup of the character group associated to the mod q reduction of $J^D(\Gamma_0(pq) \cap \Gamma)$. This isomorphism played a key role in his proof of the level-lowering theorem. It also allowed him to show [Ri2] that even in the case where $D = 1$ and Γ is trivial, $J_0(pq)_{pq\text{-new}}$ and $J^{pq}(1)$ could have decidedly different structures. In particular, he constructed examples of

maximal ideals \mathfrak{m} of the Hecke algebra \mathbb{T} for which $J_0(pq)_{pq-\text{new}}[\mathfrak{m}]$ had dimension two yet $J^{pq}(1)[\mathfrak{m}]$ had dimension four. In particular, this would happen whenever the $(\bmod l)$ Galois representation $\rho_{\mathfrak{m}}$ associated to \mathfrak{m} was unramified at q and satisfied $\rho_{\mathfrak{m}}(\text{Frob}_q) = \pm 1$. Since any isogeny between the two varieties would have to have nontrivial kernel at all such \mathfrak{m} , any description of the kernel of such an isogeny would necessarily contain information about which maximal ideals \mathfrak{m} of \mathbb{T} had the above property.

In this paper we study isogenies between the varieties described above using an approach first suggested by Mazur. Instead of attempting to produce isogenies between the two varieties directly, we study the entire category of abelian varieties with an action of the Hecke-algebra that are Hecke-equivariantly isogenous to $J^{Dpq}(\Gamma)$.

More precisely, fix a squarefree integer N , prime to the level M of Γ , and for each D dividing N and divisible by an even number of primes, let $J^D = J^D(\Gamma_0(N/D) \cap \Gamma)_{\frac{N}{D}-\text{new}}$. Let \mathbb{T} be the subalgebra of the endomorphism ring of some such J^D generated by the Hecke operators T_l for l prime and the diamond bracket operators $\langle d \rangle$. Repeated applications of the Jacquet–Langlands correspondence show that \mathbb{T} is independent (up to canonical isomorphism) of the choice of J^D used to define it, and therefore acts simultaneously on all the J^D . Moreover, the abelian varieties J^D are all \mathbb{T} -equivariantly isogenous. We let \mathcal{A} denote the category of abelian varieties with a \mathbb{T} -action that are \mathbb{T} -equivariantly isogenous to J^1 , and develop a formalism that provides a concrete description of \mathcal{A} in terms of \mathbb{T} -modules.

In particular, we show that if one is willing to work “up to support on S ”, where S is the set of Eisenstein primes of \mathbb{T} , one can classify all abelian varieties in \mathcal{A} in terms of isomorphism classes of \mathbb{T} -modules. Specifically, to any abelian variety J isogenous to J^1 one can associate a certain \mathbb{T} -module $[J]$ that is free over \mathbb{Z} and satisfies $[J] \otimes \mathbb{Q} \cong \mathbb{T} \otimes \mathbb{Q}$. (Such modules were first introduced by Mazur [Ma], who called them “rank one” modules.) After formally inverting isogenies supported on S , and maps of \mathbb{T} -modules whose cokernels are supported on S , this functor becomes an antiequivalence of categories (Theorem 4.11). Sections 3 and 4 are devoted to making this precise. Moreover, if J and J' are isogenous to J^1 , knowing the isomorphism classes of $[J]$ and $[J']$ as \mathbb{T} -modules is often enough (Proposition 4.12) to determine the kernels of maps $J \rightarrow J'$ up to support on S . Thus computing $[J]$ and $[J']$ allows us to construct the isomorphism class of J' in terms of the isomorphism class of J . In particular one can express the Tate modules of J' at maximal ideals outside S in terms of

the Tate modules of J and the modules $[J]$ and $[J']$.

Moreover, we obtain a relationship between the character groups $X_p(J)$ of the tori coming from the mod p reductions of J for p dividing N , and the module $[J]$, for any J isogenous to J^1 (Proposition 4.14). In Section 6 we establish a key relationship between the character groups of J^D as D varies. These two results allow us to compute the modules $[J^D]$ explicitly. The key result we obtain is Theorem 6.1, in which we obtain $[J^D]$ for $D = N$ in terms of a certain module of maps between character groups.

Sections 6 and 7 are devoted to the proof of Theorem 6.1. In section 6, we introduce a condition, called *controllability*, associated to a prime p dividing N and a maximal ideal \mathfrak{m} of \mathbb{T} . Controllability of \mathfrak{m} at p implies that the character group of a certain abelian variety isogenous to J^D becomes free of rank one after completing at \mathfrak{m} (c.f. Lemma 6.5); in section 6 we exploit this fact to show that Theorem 6.1 holds locally at any maximal ideal \mathfrak{m} that is controllable at some prime dividing D .

It remains to handle those maximal ideals that are not controllable at any primes dividing D , which we do in Section 7. Although such ideals are not at all common, they are much more difficult to handle, and the arguments presented here are of a very technical sort. (The reader who is primarily interested in our results, or who is willing to take this extreme case of Theorem 6.1 on faith, is encouraged to read Section 8 first; once Theorem 6.1 is established we have no further need of the results of Section 7.) Our basic strategy for dealing with these maximal ideals is a level-raising argument. In particular, we fix such a maximal ideal \mathfrak{m} , and introduce two primes q_1 and q_2 to the level, in such a way that \mathfrak{m} is controllable at both q_1 and q_2 . We establish an analogue of Theorem 6.1 in this new setting, using controllability at q_1 and q_2 , and then show that it implies the desired result.

In Section 8 we explore the consequences of Theorem 6.1. In particular, we apply Theorem 6.1 along with Proposition 5.7 to obtain several expressions for $[J^D]$ in terms of $[J^{D'}]$ whenever D' divides D (Theorem 8.5, Theorem 8.7 and Corollary 8.14). In particular, if p is a divisor of D' , we find that the \mathbb{T} -modules $\mathrm{Hom}(J^{D'}, J^D)$ and $\mathrm{Hom}(X_p(J^D), X_p(J^{D'}))$ are isomorphic up to support on S , via the natural map that takes a morphism of varieties to the map it induces on character groups. Via Proposition 4.12, this allows us to determine the kernels of all morphisms $J^D \rightarrow J^{D'}$, up to support on S . We give several different descriptions of these kernels (Corollaries 8.6, 8.8 and 8.15) with varying degrees of explicitness. The latter two expressions, for instance, are in terms of character

groups that can be computed algorithmically [Ko]. The upshot of this is that given the isomorphism class of J^1 , one can explicitly compute the isomorphism class of J^D for any D , up to support on S .

This recipe for constructing J^D from J^1 allows us to compute many invariants of J^D in terms of J^1 . In particular one can compute the \mathfrak{m} -adic Tate modules of J^D in this fashion, for any \mathfrak{m} outside S . In particular one obtains expressions for the dimension of $J^D[\mathfrak{m}]$ in this manner. In this fashion, we obtain a bound on the dimension of $J^D[\mathfrak{m}]$ in terms of the number of primes dividing D at which \mathfrak{m} is not controllable, a significant strengthening of an earlier result of L. Yang [Ya]. An important special case of this bound is when \mathfrak{m} is controllable at every prime and $J^1[\mathfrak{m}]$ has dimension two; in this case $J^D[\mathfrak{m}]$ has dimension two as well—that is, we have “multiplicity one” for J^D at \mathfrak{m} .

2. Notation, background, and conventions

Before we begin, we fix notation, explicitly state the background results referred to in the introduction, and establish some conventions that will be in use throughout the paper.

Definition 2.1: Let N be an integer, and H a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. We define $\Gamma_H(N)$ to be the subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$ consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ for which $c \equiv 0 \pmod{N}$, d ‘lies in’ $H \pmod{N}$, and a and b are arbitrary. By a congruence subgroup of level N , we mean a subgroup of the form $\Gamma_H(N)$ for some H .

Let D be a squarefree integer divisible by an even number of primes, and let B be a quaternion algebra of discriminant D . For each prime p not dividing D , we fix an isomorphism $B \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$. Then if N is an integer prime to D , and Γ a congruence subgroup of level N , the isomorphisms above determine a subgroup of $B^\times(\mathbb{A}_{\mathbb{Q}}^f)$. We denote by $X^D(\Gamma)$ the Shimura curve associated to the quaternion algebra B with level structure given by this subgroup. This curve is defined over \mathbb{Q} , and up to isomorphism, it is independent of the choice of identifications made above. We let $J^D(\Gamma)$ denote the Jacobian of $X^D(\Gamma)$. (If $D = 1$ we take $X^D(\Gamma)$ to be the modular curve with Γ -level structure, instead.)

Let T be a squarefree integer dividing N , and suppose that Γ has the form $\Gamma_0(T) \cap \Gamma'$, where Γ' is a congruence subgroup of level N/T and $\Gamma_0(T) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ is the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which c is divisible by T . Let p divide T . Then there are two natural degeneracy maps:

$$\alpha_p, \beta_p: X^D(\Gamma) \rightarrow X^D(\Gamma_0(T/p) \cap \Gamma').$$

These induce maps

$$J^D(\Gamma) \rightarrow J^D(\Gamma_0(T/p) \cap \Gamma')^2 \quad \text{and} \quad J^D(\Gamma_0(T/p) \cap \Gamma')^2 \rightarrow J^D(\Gamma)$$

by Albanese and Picard functoriality, respectively. We let $J^D(\Gamma)_{p\text{-old}}$ denote the image of the latter map, and $J^D(\Gamma)_{p\text{-new}}$ denote the connected component of the identity in the kernel of the former map. Both are abelian subvarieties of $J^D(\Gamma)$, stable under the action of the Hecke algebra. If U is a divisor of T , we let $J^D(\Gamma)_{U\text{-old}}$ (resp. $J^D(\Gamma)_{U\text{-new}}$) denote the connected component of the identity in the intersections of the varieties $J^D(\Gamma)_{p\text{-old}}$ (resp. $J^D(\Gamma)_{p\text{-new}}$) for p dividing U .

We denote by $\mathbb{T}^D(\Gamma)$ the subalgebra of $\text{End}(J^D(\Gamma))$ generated by the Hecke operators T_l for l prime, and the diamond bracket operators $\langle d \rangle$ for d in $(\mathbb{Z}/N\mathbb{Z})^\times$. (When l divides N we take the T_l to be defined as in [Ril].) The algebra $\mathbb{T}^D(\Gamma)$ is a finite, flat \mathbb{Z} -algebra. If $\Gamma = \Gamma_0(T) \cap \Gamma'$ as above, and U divides \mathbb{T} , then $\mathbb{T}^D(\Gamma)$ acts on $J^D(\Gamma)_{U\text{-new}}$ (resp. $J^D(\Gamma)_{U\text{-old}}$) through a finite quotient that is also flat over \mathbb{Z} . We denote this quotient by $\mathbb{T}^D(\Gamma)^{U\text{-new}}$ (resp. $\mathbb{T}^D(\Gamma)^{U\text{-old}}$).

If J is an abelian variety with a $\mathbb{T}^D(\Gamma)$ -action, then this action allows us to speak of the U -new subvarieties and quotients of J as well. We take the U -new subvariety of J to be the connected component of the subvariety of J annihilated by the kernel I of the map $\mathbb{T}^D(\Gamma) \rightarrow \mathbb{T}^D(\Gamma)^{U\text{-new}}$, and the U -new quotient of J to be J/IJ . The U -old subvariety and quotient of J are constructed in a similar manner. It is straightforward to check that this construction agrees with the previous definition of $J^D(\Gamma)_{U\text{-old}}$ and $J^D(\Gamma)_{U\text{-new}}$.

We have the following relationship between the subvarieties and quotients defined above and duality of abelian varieties:

PROPOSITION 2.2: *Suppose J is isogenous to $J^D(\Gamma)$. Then have $(J_{U\text{-new}})^\vee \cong (J^\vee)^{U\text{-new}}$.*

Proof: Consider the exact sequence

$$0 \rightarrow J_{U\text{-new}} \rightarrow J \rightarrow A \rightarrow 0.$$

Taking duals, we obtain an exact sequence

$$0 \rightarrow A^\vee \rightarrow J^\vee \rightarrow (J_{U\text{-new}})^\vee \rightarrow 0.$$

(Here the action of \mathbb{T} on J^\vee is induced by duality of abelian varieties.) It thus suffices to show that we can identify A^\vee with IJ^\vee , where I is the kernel of

the map $\mathbb{T}^D(\Gamma) \rightarrow \mathbb{T}^D(\Gamma)^{U-\text{new}}$. Since I annihilates $(J_{U-\text{new}})^\vee$, we must have $IJ^\vee \subset A^\vee$. Counting ranks of abelian varieties, we find that the two must be equal. ■

Fix a particular D and N , a congruence subgroup Γ' of level N , and two distinct primes p and q that do not divide DN . Let $\Gamma = \Gamma' \cap \Gamma_0(pq)$. The tangent spaces $S_2^D(\Gamma)_{pq-\text{new}}$ and $S_2^{Dpq}(\Gamma')$ of $J^D(\Gamma)_{pq-\text{new}}$ and $J^{Dpq}(\Gamma')$ over \mathbb{Q} , respectively, are modules for $\mathbb{T}^D(\Gamma)^{pq-\text{new}}$ and $\mathbb{T}^{Dpq}(\Gamma')$.

THEOREM 2.3 (Jacquet–Langlands, Ribet): *There is a natural isomorphism $\mathbb{T}^D(\Gamma)^{pq-\text{new}} \rightarrow \mathbb{T}^{Dpq}(\Gamma')$ that sends an operator T_l in one algebra to the corresponding operator T_l in the other (and similarly for $\langle d \rangle$.) Moreover, the $\mathbb{T}^{Dpq}(\Gamma')$ -modules $S_2^D(\Gamma)_{pq-\text{new}}$ and $S_2^{Dpq}(\Gamma')$ are isomorphic.*

Proof: Jacquet and Langlands [J-L] showed this for the subalgebras of $\mathbb{T}^D(\Gamma)^{pq-\text{new}}$ and $\mathbb{T}^{Dpq}(\Gamma')$ generated by the diamond bracket operators and the Hecke operators T_l for l prime to $DpqN$. The results of Ribet [Ri1] (particularly Theorems 3.20 and 4.1) imply that it holds for the full Hecke algebra. ■

COROLLARY 2.4: *There is a \mathbb{T} -equivariant isogeny:*

$$J^D(\Gamma)_{pq-\text{new}} \rightarrow J^{Dpq}(\Gamma').$$

Proof: Let $\mathbb{T} = \mathbb{T}^{Dpq}(\Gamma')$, and V_l and V'_l the l -adic Tate modules of $J = J^D(\Gamma)_{pq-\text{new}}$ and $J' = J^{Dpq}(\Gamma')$, respectively. Then $V_l \otimes \mathbb{Q}$ and $V'_l \otimes \mathbb{Q}$ are both free of rank two over $\mathbb{T} \otimes \mathbb{Q}_l$, and for any l prime to $DpqN$, the trace and determinant of Frob_l on both modules is given by T_l and $l\langle l \rangle$, respectively. It follows that $V_l \otimes \mathbb{Q}$ and $V'_l \otimes \mathbb{Q}$ are \mathbb{T} -equivariantly isomorphic as $G_{\mathbb{Q}}$ -modules. By [Fa], Theorem 4, J and J' are \mathbb{T} -equivariantly isogenous. ■

We finish this section with some notation and conventions.

We will often be considering modules over a finite, flat \mathbb{Z} -algebra \mathbb{T} . If M is a \mathbb{T} -module, and \mathfrak{m} is a maximal ideal of \mathbb{T} , then $M_{\mathfrak{m}}$ denotes the *completion* of M at \mathfrak{m} . If M is free as a \mathbb{Z} -module, then M^* denotes the \mathbb{Z} -dual $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, with the \mathbb{T} -module structure induced from M . If M is free as a \mathbb{Z}_l -module for some l , on the other hand, then by abuse of notation we write M^* for $\text{Hom}_{\mathbb{Z}_l}(M, \mathbb{Z}_l)$. The advantage of this notation is that $(M_{\mathfrak{m}})^*$ is naturally isomorphic to $(M^*)_{\mathfrak{m}}$ for any maximal ideal \mathfrak{m} of \mathbb{T} .

The \mathbb{T} -modules we are most interested in are all torsion free over \mathbb{Z} . Unfortunately, the tensor product of two \mathbb{T} -modules without \mathbb{Z} -torsion often has

\mathbb{Z} -torsion, which we wish to ignore. Therefore we adopt the convention that if M and M' are torsion free \mathbb{T} -modules or $\mathbb{T}_{\mathfrak{m}}$ -modules, then $M \otimes M'$ denotes their tensor product (over \mathbb{T} or $\mathbb{T}_{\mathfrak{m}}$) *modulo \mathbb{Z} -torsion*. It can be verified that this operation satisfies the universal property of the tensor product in the category of \mathbb{T} -modules without \mathbb{Z} -torsion. Note that if I and I' are ideals of \mathbb{T} , then $I \otimes I'$ is isomorphic to II' under this convention.

We will find the following observation about these two operations to be useful in what follows.

LEMMA 2.5: *Let M and N be finitely generated, torsion free \mathbb{T} -modules. We have natural isomorphisms:*

1. $M^* \cong \mathrm{Hom}_{\mathbb{T}}(M, \mathbb{T}^*)$
2. $(M \otimes N)^* \cong \mathrm{Hom}_{\mathbb{T}}(M, N^*)$
3. $\mathrm{Hom}_{\mathbb{T}}(M, N)^* \cong M \otimes N^*$.

Proof:

1. We have $M^* \cong \mathrm{Hom}_{\mathbb{T}}(\mathbb{T}, M^*)$. Taking \mathbb{Z} -duals, we find that the latter is naturally isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M, \mathbb{T}^*)$.
2. By (1), $(M \otimes N)^*$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M \otimes N, \mathbb{T}^*)$, which in turn is canonically isomorphic to $\mathrm{Hom}_{\mathbb{T}}(M, \mathrm{Hom}_{\mathbb{T}}(N, \mathbb{T}^*))$, and the result follows.
3. This follows from (2) by replacing N with N^* and dualizing. ■

3. Morphisms “up to support on S ”

In what follows, we will often find it convenient to work in categories of abelian varieties in which certain morphisms have been formally inverted. In particular, we will fix a finite flat \mathbb{Z} -algebra \mathbb{T} and an abelian variety J/\mathbb{Q} on which \mathbb{T} acts faithfully.

Definition 3.1: Let \mathcal{A} be the category whose objects consist of abelian varieties J' with an action of \mathbb{T} , such that there exists a \mathbb{T} -equivariant isogeny $\phi: J \rightarrow J'$ defined over \mathbb{Q} , and whose morphisms are \mathbb{T} -equivariant maps of abelian varieties defined over \mathbb{Q} .

We are interested in the behavior of maps in this category up to support on a certain finite set S of maximal ideals of \mathbb{T} . In order to exclude phenomena that happen solely at maximal ideals inside S , we wish to formally invert morphisms in \mathcal{A} that are isomorphisms at all \mathfrak{m} outside S . In order to make this precise, we invoke the following

LEMMA 3.2: *Let S be a finite set of maximal ideals of \mathbb{T} . Then there is an element σ_S of \mathbb{T} such that for any maximal ideal \mathfrak{m} of \mathbb{T} , $\sigma_S \in \mathfrak{m}$ if and only if $\mathfrak{m} \in S$. In particular, $U = \text{Spec } \mathbb{T} - S$ is affine, and equals to $\text{Spec } \mathbb{T}[1/\sigma_S]$.*

Proof: It suffices to show this when \mathbb{T} is reduced, as if σ_S^{red} is an element of \mathbb{T}^{red} (the reduced quotient of \mathbb{T}) with the desired property, then any lift of σ_S^{red} to \mathbb{T} will have the desired property.

Suppose \mathbb{T} is reduced, and let $\hat{\mathbb{T}}$ be its normalization. Then $\hat{\mathbb{T}}$ is a product of maximal orders in number fields, and so its Picard group is finite. In particular, for each maximal ideal \mathfrak{m} of \mathbb{T} , we can find an element $\hat{\sigma}_{\mathfrak{m}}$ of $\hat{\mathbb{T}}$ such that $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathfrak{m}\hat{\mathbb{T}}$ but does not lie in $\mathfrak{m}'\hat{\mathbb{T}}$ for any \mathfrak{m}' different from \mathfrak{m} .

LEMMA 3.3: *For n sufficiently large, $\hat{\sigma}_{\mathfrak{m}}^n$ lies in \mathbb{T} .*

Proof: Let N be the index of \mathbb{T} in $\hat{\mathbb{T}}$. It suffices to check this statement locally at each maximal ideal of \mathbb{T} containing N . Since $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathfrak{m}\hat{\mathbb{T}}$, for some n $\sigma_{\mathfrak{m}}^n$ lies in $N\hat{\mathbb{T}}_{\mathfrak{m}}$ and therefore lies in $\mathbb{T}_{\mathfrak{m}}$. Fix a maximal ideal \mathfrak{m}' of \mathbb{T} distinct from \mathfrak{m} but containing N . Then $\hat{\sigma}_{\mathfrak{m}}$ lies in $\mathbb{T}_{\mathfrak{m}'}^{\times}$. Let n be the order of $(\mathbb{T}_{\mathfrak{m}'}/N\mathbb{T}_{\mathfrak{m}'})^{\times}$. Then $\hat{\sigma}_{\mathfrak{m}}^n \in 1 + N\mathbb{T}_{\mathfrak{m}'} \subset \mathbb{T}_{\mathfrak{m}'}$, as required. ■

We now resume the proof of Lemma 3.2. Let $\sigma_{\mathfrak{m}} = \hat{\sigma}_{\mathfrak{m}}^n$. Then $\sigma_{\mathfrak{m}}$ lies in \mathfrak{m} , but does not lie in \mathfrak{m}' for any \mathfrak{m}' distinct from \mathfrak{m} . Taking σ_S equal to the product of the $\sigma_{\mathfrak{m}}$ gives us a σ_S with the desired properties. ■

Let $\mathbb{T}_S = \mathbb{T}[1/\sigma_S]$. Define \mathcal{A}_S to be the category whose objects are the objects of \mathcal{A} , but whose morphisms are given by

$$\text{Hom}_{\mathcal{A}_S}(A_1, A_2) = \text{Hom}_{\mathcal{A}}(A_1, A_2) \otimes_{\mathbb{T}} \mathbb{T}_S.$$

This has the desired effect of ignoring behavior supported on S , as any morphism whose kernel is supported on S factors through a sufficiently high power of σ_S , and hence is an isomorphism in \mathcal{A}_S . We will make use of this construction in the next section.

Observe that since σ_S is a unit in $\mathbb{T}_{\mathfrak{m}}$ for any \mathfrak{m} outside S , any morphism $A_1 \rightarrow A_2$ in \mathcal{A}_S induces a well defined map $A_1[\mathfrak{m}^{\infty}] \rightarrow A_2[\mathfrak{m}^{\infty}]$ that sends $x \in A_1[\mathfrak{m}^{\infty}]$ to $\sigma_S^{-n}(\sigma_S^n f)(x)$, for n sufficiently large to make σ_S^n a morphism in \mathcal{A} .

It will be convenient to speak of the “kernel” of a map in \mathcal{A}_S , even though strictly speaking this notion is not well defined. It is possible to make it precise “up to support on S ”, however; for any morphism $f: A_1 \rightarrow A_2$ in \mathcal{A}_S , we can

fix an n such that $\sigma_S^n f$ is a morphism in \mathcal{A} , and take $\ker_S f = \ker \sigma_S^n f$. This is independent of n up to a finite group supported on S .

Definition 3.4: Let B be a subvariety of A_1 that is stable under \mathbb{T} and closed under addition and inverses, and $f: A_1 \rightarrow A_2$ a morphism in \mathcal{A}_S . Then $\ker_S f = B$ “**up to support on S** ” if for some n , $\sigma_S^n f$ is a morphism in \mathcal{A} , $\ker \sigma_S^n f$ contains B , and the group $(\ker \sigma_S^n f)/B$ is finite and supported on S . (Equivalently, $\ker_S f = B$ “up to support on S ” if $B[\mathfrak{m}^\infty] = \ker f: A_1[\mathfrak{m}^\infty] \rightarrow A_2[\mathfrak{m}^\infty]$ for all \mathfrak{m} outside S .)

4. The isogeny class of an abelian variety with \mathbb{T} -action

We now begin the program described in the introduction, in which we study the category of abelian varieties isogenous to $J^D(\Gamma)$ for some D and Γ . However, our techniques apply in considerably more generality, so we work in the following setting; let J be an abelian variety defined over \mathbb{Q} , equipped with the action of a finite, flat \mathbb{Z} -algebra \mathbb{T} , in which the action of any element σ of \mathbb{T} on J is defined over \mathbb{Q} .

If \mathfrak{m} is a maximal ideal of \mathbb{T} of residue characteristic l , let $T_{\mathfrak{m}}J$ denote the \mathfrak{m} -adic **contravariant** Tate module of J at \mathfrak{m} , that is, the inverse limit of $J[\mathfrak{m}^k]^\vee$ over k , where $(-)^\vee$ denotes Cartier duality. Thus $T_{\mathfrak{m}}J$ is naturally isomorphic to $\text{Hom}(J[\mathfrak{m}^\infty], \mu_{l^\infty})$. If $\alpha \in T_{\mathfrak{m}}J$ and $x \in J[\mathfrak{m}^\infty]$, we denote by $\alpha(x)$ the root of unity obtained by evaluating the element of $\text{Hom}(J[\mathfrak{m}^\infty], \mu_{l^\infty})$ that corresponds to α at the element x . For a morphism $f: A_1 \rightarrow A_2$ of abelian varieties with \mathbb{T} -action, we let $f_{\mathfrak{m}}$ denote the induced map $T_{\mathfrak{m}}A_2 \rightarrow T_{\mathfrak{m}}A_1$.

The action of \mathbb{T} on J induces an action of $\mathbb{T} \otimes \mathbb{Q}$ on the singular cohomology $H^1(J(\mathbb{C}), \mathbb{Q})$ of J ; we assume that this makes $H^1(J(\mathbb{C}), \mathbb{Q})$ into a free $\mathbb{T} \otimes \mathbb{Q}$ -module, of rank two. In particular, this means that the action of \mathbb{T} on J is **faithful**.

Since there are natural isomorphisms, $H^1(J(\mathbb{C}), \mathbb{Z})_{\mathfrak{m}} \cong T_{\mathfrak{m}}J$, $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ is free of rank two as a $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$ -module. The action of $G_{\mathbb{Q}}$ on $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ gives us a two-dimensional representation $\rho_{\mathfrak{m}}$ of $G_{\mathbb{Q}}$ over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$. We impose the further hypothesis that $\text{Tr } \rho_{\mathfrak{m}}(g)$ and $\det \rho_{\mathfrak{m}}(g)$ (a priori elements of $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$) lie in $\mathbb{T}_{\mathfrak{m}}$ for all \mathfrak{m} and g . We further assume that the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the elements $\text{Tr } \rho_{\mathfrak{m}}(g)$ and $\det \rho_{\mathfrak{m}}(g)$ as g varies is **reduced** for all \mathfrak{m} .

Finally, we assume that there exists a \mathbb{T} -equivariant map $\psi: J \rightarrow J^\vee$, defined over a finite extension K of \mathbb{Q} , and a character χ of $\text{Gal}(K/\mathbb{Q})$, taking values in \mathbb{T}^\times , such that ψ has finite kernel, and for all $\sigma \in \text{Gal}(K/\mathbb{Q})$, we have $\sigma\psi = \chi(\sigma)\psi\sigma$.

Remark 4.1: The motivation for these assumptions comes from the example in which $J = J^D(\Gamma)$ for some discriminant D and congruence subgroup Γ of level N , and \mathbb{T} is the Hecke algebra $\mathbb{T}^D(\Gamma)$. In this case it can be shown (see for instance Carayol [Ca]) that $H^1(J(\mathbb{C}), \mathbb{Q})$ is free of rank two over $\mathbb{T} \otimes \mathbb{Q}$. Moreover, for a prime l not dividing ND , $\text{Tr } \rho_{\mathfrak{m}}(\text{Frob}_l)$ is equal to $T_l \in \mathbb{T}$, and $\det \rho_{\mathfrak{m}}(\text{Frob}_l) = l \langle l \rangle$. The subalgebra of \mathbb{T} generated by $\langle l \rangle$ and T_l for l prime to ND is indeed reduced, since it is well-known that there exists a basis of simultaneous eigenforms for all such T_l . Since the elements Frob_l for l prime to N are dense in $G_{\mathbb{Q}}$ and $\mathbb{T}_{\mathfrak{m}}$ is complete, the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the traces and determinants of elements of $G_{\mathbb{Q}}$ is generated by the aforementioned $\langle l \rangle$ and T_l , and is therefore reduced. Finally, the map ψ described above is constructed in Lemma 5.5. It follows that the hypotheses of this section are satisfied in this case.

The existence of the map ψ described above has strong consequences for the structure of \mathbb{T} . In particular, ψ induces an isomorphism $T_{\mathfrak{m}} J \otimes \mathbb{Q} \rightarrow (T_{\mathfrak{m}} J)^* \otimes \mathbb{Q}$. Since the former is free of rank two over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$, and the latter is isomorphic to $(\mathbb{T}_{\mathfrak{m}}^* \otimes \mathbb{Q})^2$, it follows that $\mathbb{T}_{\mathfrak{m}}^* \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$. In particular, $\mathbb{T} \otimes \mathbb{Q}$ is a zero-dimensional Gorenstein ring.

Following the analysis of the Tate modules of modular Jacobians in [Ca], we fix a maximal ideal \mathfrak{m} of \mathbb{T} , and let $\mathbb{T}'_{\mathfrak{m}}$ denote the subalgebra of $\mathbb{T}_{\mathfrak{m}}$ generated by the traces and determinants of $\rho_{\mathfrak{m}}(g)$ for $g \in G_{\mathbb{Q}}$. By our hypothesis, this is a reduced subalgebra of $\mathbb{T}_{\mathfrak{m}}$. We can thus consider the normalization $\hat{\mathbb{T}}'_{\mathfrak{m}}$ of $\mathbb{T}'_{\mathfrak{m}}$; this will be a product of discrete valuation rings \mathcal{O}_i , with corresponding fields of fractions K_i .

For each i , $T_{\mathfrak{m}} J \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$ is free of rank two over the Artinian K_i -algebra $\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$; we consider the representation $\rho_{\mathfrak{m},i}$ of $G_{\mathbb{Q}}$ over $\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} K_i$ obtained in this manner. Since the traces and determinants of $\rho_{\mathfrak{m}}$ lie in $\mathbb{T}'_{\mathfrak{m}}$, the traces and determinants of $\rho_{\mathfrak{m},i}$ lie in \mathcal{O}_i for all i . Thus $\rho_{\mathfrak{m},i}$ can be considered as a representation of $G_{\mathbb{Q}}$ over \mathcal{O}_i .

Let k_i denote the residue field of \mathcal{O}_i ; then we may consider the mod \mathfrak{m} reduction $\bar{\rho}_{\mathfrak{m},i}: G_{\mathbb{Q}} \rightarrow \text{GL}_2(k_i)$ of $\rho_{\mathfrak{m},i}$. For $g \in G$, $\text{Tr } \bar{\rho}_{\mathfrak{m},i}(g)$ is equal to the reduction of $\text{Tr } \rho_{\mathfrak{m}}(g)$ modulo \mathfrak{m} , and is therefore independent of i . Thus the semisimplification of $\bar{\rho}_{\mathfrak{m},i}$ is independent of i .

With this definition, we make one final assumption on J , namely, that for all but finitely many \mathfrak{m} , the representations $\bar{\rho}_{\mathfrak{m},i}$ are absolutely irreducible. For such \mathfrak{m} , $\bar{\rho}_{\mathfrak{m},i}$ is independent of i and we denote it simply by $\bar{\rho}_{\mathfrak{m}}$.

Remark 4.2: It should be pointed out here that in the case in which J is a modular Jacobian, the $\bar{\rho}_{\mathfrak{m}}$ constructed above is the Cartier dual of the representation usually called $\bar{\rho}_{\mathfrak{m}}$ in the literature. (This is because the latter is defined in terms of the \mathfrak{m} -torsion of J , whereas the representations we consider are defined in terms of $T_{\mathfrak{m}}J$ instead.)

Fix a finite set S of maximal ideals of \mathbb{T} , containing all \mathfrak{m} for which one (and hence all) of the $\bar{\rho}_{\mathfrak{m},i}$ is not absolutely irreducible. The following result is a straightforward generalization of [Ca], Theorem 3:

PROPOSITION 4.3: *For all \mathfrak{m} outside S , there exists a $G_{\mathbb{Q}}$ -module $V_{\mathfrak{m}}$, free of rank two over $\mathbb{T}_{\mathfrak{m}}$, such that $V_{\mathfrak{m}} \otimes \mathbb{Q}$ and $T_{\mathfrak{m}}J \otimes \mathbb{Q}$ are isomorphic as $G_{\mathbb{Q}}$ -modules.*

Proof: Consider the action of $G_{\mathbb{Q}}$ on $\hat{V}'_{\mathfrak{m}} = (\hat{\mathbb{T}}'_{\mathfrak{m}})^2$ in which $G_{\mathbb{Q}}$ acts on each factor $(\mathcal{O}_i)^2$ of $(\hat{\mathbb{T}}'_{\mathfrak{m}})^2$ by $\rho_{\mathfrak{m},i}$. This gives a representation of $G_{\mathbb{Q}}$ in $\mathrm{GL}_2(\hat{\mathbb{T}}'_{\mathfrak{m}})$ whose character agrees with $\rho_{\mathfrak{m}}$. By [Ca], Theorem 2, this representation is realizable over $\mathbb{T}'_{\mathfrak{m}}$, i.e., there is a $G_{\mathbb{Q}}$ -module $V'_{\mathfrak{m}}$ free of rank two over $\mathbb{T}'_{\mathfrak{m}}$ such that the character of the $G_{\mathbb{Q}}$ -action agrees with that of $\rho_{\mathfrak{m}}$. Then $V_{\mathfrak{m}} = V'_{\mathfrak{m}} \otimes_{\mathbb{T}'_{\mathfrak{m}}} \mathbb{T}_{\mathfrak{m}}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$ and $V_{\mathfrak{m}} \otimes \mathbb{Q}$ is equivalent to $\rho_{\mathfrak{m}}$ by [Ca], Theorem 1. ■

Our goal is to study \mathbb{T} -equivariant isogenies of J that are defined over \mathbb{Q} , up to support on S . The first step in this study is to understand the structure of $T_{\mathfrak{m}}J$ for \mathfrak{m} outside S .

LEMMA 4.4: *For all but finitely many maximal ideals \mathfrak{m} of \mathbb{T} , $T_{\mathfrak{m}}J$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$.*

Proof: Since we have assumed that $H^1(J(\mathbb{C}), \mathbb{Q})$ is free of rank two over $\mathbb{T} \otimes \mathbb{Q}$, we can choose two elements of $H^1(J(\mathbb{C}), \mathbb{Z})$ that generate $H^1(J(\mathbb{C}), \mathbb{Q})$ over $\mathbb{T} \otimes \mathbb{Q}$. These two elements give us a map $\mathbb{T}^2 \rightarrow H^1(J(\mathbb{C}), \mathbb{Z})$, with finite cokernel. Localizing this map at any \mathfrak{m} on which the cokernel has no support gives the desired isomorphism $\mathbb{T}_{\mathfrak{m}}^2 \cong T_{\mathfrak{m}}J$. ■

We will see (Lemma 4.6, below) that the Tate modules at such \mathfrak{m} are particularly well-behaved. It will thus be convenient for us to replace J with an isogenous abelian variety, which we call J^{\min} , for which $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$ for all \mathfrak{m} outside S . The following lemma allows us to do so.

LEMMA 4.5: *There exists an abelian variety J^{\min} , with a \mathbb{T} -action satisfying the above hypotheses, such that J and J^{\min} are \mathbb{T} -equivariantly isogenous (over \mathbb{Q}), and $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$ for all \mathfrak{m} outside S .*

Proof: Let S' be the set of \mathfrak{m} outside S at which $T_{\mathfrak{m}}J$ is not free of rank two; then S' is finite. For each \mathfrak{m} in S , fix an isomorphism $V_{\mathfrak{m}} \otimes \mathbb{Q} \cong T_{\mathfrak{m}}J \otimes \mathbb{Q}$. Multiplying this isomorphism by a sufficiently large $n \in \mathbb{Z}$, we may assume that it maps $V_{\mathfrak{m}}$ into $T_{\mathfrak{m}}J$, with finite index. Let $W_{\mathfrak{m}} \subset J[\mathfrak{m}^{\infty}]$ be the subset annihilated by the image of $V_{\mathfrak{m}}$ in $T_{\mathfrak{m}}J$, under the natural pairing $T_{\mathfrak{m}}J \times J[\mathfrak{m}^{\infty}] \rightarrow \mu_{p^{\infty}}$. If we take J^{\min} to be the quotient of J by the sum of the $W_{\mathfrak{m}}$, we find that the map $T_{\mathfrak{m}}J^{\min} \rightarrow T_{\mathfrak{m}}J$ identifies $T_{\mathfrak{m}}J^{\min}$ with the image of $V_{\mathfrak{m}}$ for all \mathfrak{m} in S' , and is an isomorphism for \mathfrak{m} outside S' ; in either case $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. ■

We fix, for the remainder of this section, a J^{\min} as above.

LEMMA 4.6: *Let M be a $G_{\mathbb{Q}}$ -stable submodule of $T_{\mathfrak{m}}J^{\min}$, of finite index. If \mathfrak{m} lies outside S , then $M = IT_{\mathfrak{m}}J^{\min}$ for some ideal I of \mathbb{T} .*

Proof: We use induction on the length of a maximal $G_{\mathbb{Q}}$ -stable filtration of $T_{\mathfrak{m}}J^{\min}/M$. If this length is zero, then I is the unit ideal and the result is clear. In general, let $M = M_n \subset M_{n-1} \subset \cdots \subset M_0 = T_{\mathfrak{m}}J^{\min}$ be a maximal $G_{\mathbb{Q}}$ -stable chain of submodules. By induction we may assume that $M_{n-1} = I'T_{\mathfrak{m}}J^{\min}$ for some I' .

Consider $\mathfrak{m}M_{n-1} + M$. This is $G_{\mathbb{Q}}$ -stable, contains M , and is contained in M_{n-1} . It is thus equal to either M or M_{n-1} . Suppose it were equal to M_{n-1} . Then $M/\mathfrak{m}M_{n-1}$ would equal $M_{n-1}/\mathfrak{m}M_{n-1}$, which is impossible since then M would contain a set of generators for M_{n-1} by Nakayama's lemma. Thus $\mathfrak{m}M_{n-1} + M = M$, so M contains $\mathfrak{m}M_{n-1}$.

Let $V_{\bar{\rho}_{\mathfrak{m}}}$ be a two-dimensional $(\mathbb{T}/\mathfrak{m})$ vector space on which $G_{\mathbb{Q}}$ acts via $\bar{\rho}_{\mathfrak{m}}$. The module $M_{n-1}/\mathfrak{m}M_{n-1}$ is $G_{\mathbb{Q}}$ -equivariantly isomorphic to $(I'/\mathfrak{m}I') \otimes_{\mathbb{T}/\mathfrak{m}} V_{\bar{\rho}_{\mathfrak{m}}}$, where $G_{\mathbb{Q}}$ acts trivially on $I'/\mathfrak{m}I'$. Let V be the image of M in $M_{n-1}/\mathfrak{m}M_{n-1}$. Since V is $G_{\mathbb{Q}}$ -invariant, and $V_{\bar{\rho}_{\mathfrak{m}}}$ is irreducible, V is given by $\hat{V} \otimes V_{\bar{\rho}_{\mathfrak{m}}}$ for some subspace \hat{V} of $I'/\mathfrak{m}I'$. Let I be the preimage of \hat{V} in I' . Then $IT_{\mathfrak{m}}J = M$, since both contain $\mathfrak{m}M_{n-1}$ and map to V modulo $\mathfrak{m}M_{n-1}$. ■

This result has important consequences for \mathbb{T} -equivariant maps from J^{\min} . In particular, for any abelian varieties A_1, A_2 that are \mathbb{T} -equivariantly isogenous to J over \mathbb{Q} , let $\text{Hom}(A_1, A_2)$ (resp. $\text{End}(A_1)$) denote the \mathbb{T} -module of \mathbb{T} -equivariant maps: $A_1 \rightarrow A_2$ (resp. \mathbb{T} -equivariant endomorphisms of A_1) that are defined over \mathbb{Q} . Then

COROLLARY 4.7: *The cokernel of the natural map $\mathbb{T} \rightarrow \text{End}(J^{\min})$ is supported on S .*

Proof: Fix \mathfrak{m} outside S ; we show that the natural map

$$\mathbb{T}_{\mathfrak{m}} \rightarrow \text{End}(J^{\min})_{\mathfrak{m}}$$

is an isomorphism. We have an isomorphism

$$\text{End}(J^{\min})_{\mathfrak{m}} \cong \text{End}_G(T_{\mathfrak{m}}J^{\min}),$$

by [Fa], Theorem 4. $\text{End}_G(T_{\mathfrak{m}}J^{\min})$ is clearly faithful over $\mathbb{T}_{\mathfrak{m}}$, and finitely generated. It thus suffices to show that $\text{End}_G(T_{\mathfrak{m}}J^{\min}) \otimes \mathbb{T}/\mathfrak{m}$ is 1-dimensional. Observe that $\text{End}_G(J^{\min}[\mathfrak{m}])$ is 1-dimensional, by Schur's lemma and the (absolute) irreducibility of $\bar{\rho}_{\mathfrak{m}}$. We construct an injection

$$\text{End}_G(T_{\mathfrak{m}}J^{\min}) \otimes \mathbb{T}/\mathfrak{m} \rightarrow \text{End}_G(J^{\min}[\mathfrak{m}]),$$

the existence of which immediately implies the desired result. The exact sequence

$$0 \rightarrow \mathfrak{m}T_{\mathfrak{m}}J^{\min} \rightarrow T_{\mathfrak{m}}J^{\min} \rightarrow J^{\min}[\mathfrak{m}]^{\vee} \rightarrow 0$$

induces an exact sequence:

$$0 \rightarrow \text{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}}) \rightarrow \text{End}_G(T_{\mathfrak{m}}J^{\min}) \rightarrow \text{Hom}_G(T_{\mathfrak{m}}J^{\min}, J^{\min}[\mathfrak{m}]^{\vee}),$$

and the latter is isomorphic to $\text{End}_G(J^{\min}[\mathfrak{m}])$ via Cartier duality. Now $\text{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}})$ clearly contains $\mathfrak{m} \text{End}_G(T_{\mathfrak{m}}J^{\min})$, and it suffices to show that they are equal.

Let $\phi \in \text{Hom}_G(T_{\mathfrak{m}}J^{\min}, \mathfrak{m}T_{\mathfrak{m}})$; by Lemma 4.6, the image of ϕ is isomorphic to $IT_{\mathfrak{m}}J^{\min}$ for some $I \subset \mathfrak{m}$. Since $IT_{\mathfrak{m}}J^{\min} \cong T_{\mathfrak{m}}J^{\min}$ and $T_{\mathfrak{m}}J^{\min}$ is free over $\mathbb{T}_{\mathfrak{m}}$, I is principal; say generated by σ . Thus ϕ/σ is a well defined endomorphism of $T_{\mathfrak{m}}J^{\min}$. Since $\sigma \in \mathfrak{m}$, ϕ lies in $\mathfrak{m} \text{End}_G(T_{\mathfrak{m}}J^{\min})$ as required. ■

COROLLARY 4.8: *Let $\phi: J^{\min} \rightarrow A$ be a \mathbb{T} -equivariant isogeny. Let $I = \text{Ann}(\ker \phi)$. Then $(\ker \phi)_{\mathfrak{m}} = J^{\min}[I]_{\mathfrak{m}}$ for all \mathfrak{m} outside S .*

Proof: The map ϕ induces an exact sequence

$$0 \rightarrow T_{\mathfrak{m}}A \rightarrow T_{\mathfrak{m}}J^{\min} \rightarrow (\ker \phi)_{\mathfrak{m}}^{\vee} \rightarrow 0.$$

In particular, the image of $T_{\mathfrak{m}}A$ is a \mathbb{T} -stable submodule of $T_{\mathfrak{m}}J^{\min}$. By Lemma 4.6, we can find an ideal I' of \mathbb{T} such that the image of $T_{\mathfrak{m}}A$ is $I'_{\mathfrak{m}}T_{\mathfrak{m}}J^{\min}$ for all \mathfrak{m} outside S . Then we have $(\ker \phi)_{\mathfrak{m}}^{\vee} = T_{\mathfrak{m}}J^{\min}/I'_{\mathfrak{m}}T_{\mathfrak{m}}J^{\min}$ for all \mathfrak{m}

outside S , by the above exact sequence. Thus $I'_\mathfrak{m} = I_\mathfrak{m}$ for all \mathfrak{m} outside S . Since $T_\mathfrak{m}J^{\min}/IT_\mathfrak{m}J^{\min} = J^{\min}[I]_\mathfrak{m}^\vee$, the result follows. ■

Fix a ϕ and I as above. Then for $\sigma \in I$, $\ker \phi \subset J^{\min}[I] \subset J^{\min}[\sigma]$. Thus the map $\sigma: J^{\min} \rightarrow J^{\min}$ factors through ϕ ; i.e., $\sigma = \varphi\phi$ for a unique map $\varphi: A \rightarrow J^{\min}$. Let f be the map that associates to each σ the corresponding map φ .

PROPOSITION 4.9: *The map $f: I \rightarrow \operatorname{Hom}(A, J^{\min})$ is injective, and $f_\mathfrak{m}$ is an isomorphism for all \mathfrak{m} outside S .*

Proof: The map f must be injective; since if σ induced the zero map $A \rightarrow J^{\min}$, then the action of \mathbb{T} on J^{\min} could not be faithful. It remains to show that $f_\mathfrak{m}$ is surjective for \mathfrak{m} outside S .

Let φ be an element of $\operatorname{Hom}(A, J^{\min})$. Then the composition $\varphi\phi$ lies in $\operatorname{Hom}(J^{\min}, J^{\min})$. By Corollary 4.7, the inclusion $\mathbb{T}_\mathfrak{m} \rightarrow \operatorname{Hom}(J^{\min}, J^{\min})_\mathfrak{m}$ is an isomorphism. Thus we can choose an element σ of $\mathbb{T}_\mathfrak{m}$ corresponding to $\varphi\phi$; since $J^{\min}[I]_\mathfrak{m}$ is contained in the kernel of $\varphi\phi$ by, Corollary 4.8, σ lies in $I_\mathfrak{m}$. Moreover, $f_\mathfrak{m}(\sigma) = \varphi$ by construction. ■

COROLLARY 4.10: *If A is \mathbb{T} -equivariantly isogenous to J , and \mathfrak{m} lies outside S , then the natural map*

$$T_\mathfrak{m}J^{\min} \otimes \operatorname{Hom}(A, J^{\min})_\mathfrak{m} \rightarrow T_\mathfrak{m}A$$

is an isomorphism. In particular, $T_\mathfrak{m}A$ is isomorphic to $\operatorname{Hom}(A, J^{\min})_\mathfrak{m}^2$ as a $\mathbb{T}_\mathfrak{m}$ -module.

Proof: Since $T_\mathfrak{m}J^{\min}$ is free over $\mathbb{T}_\mathfrak{m}$, $T_\mathfrak{m}J^{\min} \otimes \operatorname{Hom}(A, J^{\min})_\mathfrak{m}$ is torsion free. It thus suffices to show that the natural map is surjective, as the modules in question have the same \mathbb{Z} -rank.

Fix an isogeny $\phi: J^{\min} \rightarrow A$. Then ϕ identifies $T_\mathfrak{m}A$ with $IT_\mathfrak{m}J^{\min}$, where $I = \operatorname{Ann}(\ker \phi)$. Let $x \in T_\mathfrak{m}A$; then ϕ identifies x with an element y of $IT_\mathfrak{m}J^{\min}$; we write y as a sum of elements of the form $\sigma_i y_i$ with $\sigma_i \in I$ and $y_i \in T_\mathfrak{m}J^{\min}$. Since $\sigma_i \in I$, the map $\sigma_i: J^{\min} \rightarrow J^{\min}$ factors through ϕ , i.e., we have maps φ_i such that $\sigma_i = \varphi_i\phi$. Then $\sigma_i y_i = \varphi_i(\phi(y_i))$, so $\phi(y_i) \otimes \varphi_i$ maps to $\sigma_i y_i$ under the natural map. Thus each of the $\sigma_i y_i$ are in the image, so y is in the image as well. The second statement follows immediately. ■

Thus to any A that is \mathbb{T} -equivariantly isogenous to J we can associate the module $\operatorname{Hom}(A, J^{\min})$. This module is torsion free as a \mathbb{Z} -module, and

$\mathrm{Hom}(A, J^{\min}) \otimes \mathbb{Q}$ is free of rank one as a $\mathbb{T} \otimes \mathbb{Q}$ -module by Proposition 4.9. Following the previous section, we let \mathcal{A} be the category whose objects consist of abelian varieties A such that there exists a \mathbb{T} -equivariant isogeny $\phi : J \rightarrow A$ defined over \mathbb{Q} , and whose morphisms are \mathbb{T} -equivariant maps of abelian varieties defined over \mathbb{Q} . Then $A \mapsto \mathrm{Hom}(A, J^{\min})$ is a well defined functor $\mathcal{A} \rightarrow \mathcal{M}$, where \mathcal{M} denotes the category of finitely generated \mathbb{T} -modules M such that M is torsion free over \mathbb{Z} and $M \otimes \mathbb{Q}$ is free of rank one over $\mathbb{T} \otimes \mathbb{Q}$.

Let \mathbb{T}_S be the \mathbb{T} -algebra defined in the previous section, and consider the category \mathcal{A}_S defined therein. The functor defined above induces a functor $\mathcal{A}_S \rightarrow \mathcal{M}_S$, where \mathcal{M}_S is the category of “rank one” \mathbb{T}_S -modules, i.e., the category of finitely generated \mathbb{T}_S -modules that are torsion free over \mathbb{Z} and become free of rank one over $\mathbb{T}_S \otimes \mathbb{Q}$.

THEOREM 4.11: *The functor $A \mapsto [A] = \mathrm{Hom}(A, J^{\min})$ is an antiequivalence of categories $\mathcal{A}_S \rightarrow \mathcal{M}_S$.*

Proof: We must show that the functor is full, faithful, and essentially surjective. Given an $M \in \mathcal{M}$, we can find an injection $M \rightarrow \mathbb{T}$. Let I be its image. Then by Proposition 4.9, we have a map $I \rightarrow \mathrm{Hom}(J^{\min}/J^{\min}[I], J^{\min})$ whose cokernel is supported on S . In particular, $\mathrm{Hom}(J^{\min}/J^{\min}[I], J^{\min})$ is isomorphic to M in \mathcal{M}_S . Thus the functor is essentially surjective.

It remains to show that the functor is fully faithful. Since $\mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$ and $\mathrm{Hom}_{\mathcal{M}_S}([A_2], [A_1])$ are \mathbb{T}_S -modules, it suffices to show that the map

$$\mathrm{Hom}(A_1, A_2)_{\mathfrak{m}} \rightarrow \mathrm{Hom}([A_2], [A_1])_{\mathfrak{m}}$$

is an isomorphism for any \mathfrak{m} outside S and any $A_1, A_2 \in \mathcal{A}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(A_1, A_2)_{\mathfrak{m}} & \xrightarrow{\quad\quad\quad} & \mathrm{Hom}([A_2], [A_1])_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(T_{\mathfrak{m}}A_2, T_{\mathfrak{m}}A_1) & \longrightarrow & \mathrm{Hom}(T_{\mathfrak{m}}J^{\min} \otimes [A_2], T_{\mathfrak{m}}J^{\min} \otimes [A_1]), \end{array}$$

in which the bottom horizontal map is induced by the isomorphisms

$$[A_1] \otimes T_{\mathfrak{m}}J^{\min} \rightarrow T_{\mathfrak{m}}A_1 \quad \text{and} \quad [A_2] \otimes T_{\mathfrak{m}}J^{\min} \rightarrow T_{\mathfrak{m}}A_2$$

of Corollary 4.10. Note that each map appearing in this diagram is injective, as the modules in question have no \mathbb{Z}_l -torsion and each map becomes an isomorphism after tensoring with \mathbb{Q} . Moreover, the left-hand vertical map and the

bottom horizontal map are isomorphisms. Thus the right-hand vertical map is surjective, and hence an isomorphism. It follows that the upper horizontal map is an isomorphism as well. ■

In light of this result, we will primarily be interested in objects and morphisms in \mathcal{A}_S and \mathcal{M}_S for the remainder of this chapter. Rather than clutter our notation by writing “ $\otimes_{\mathbb{T}} \mathbb{T}_S$ ” repeatedly, we adopt the following convention: if M and N are objects of \mathcal{M} , an S -morphism: $M \rightarrow N$, is a map of \mathbb{T}_S -modules $M \otimes_{\mathbb{T}} \mathbb{T}_S \rightarrow N \otimes_{\mathbb{T}} \mathbb{T}_S$. An S -isomorphism $M \rightarrow N$ is an S -morphism that is an isomorphism in \mathcal{M}_S , and M and N are said to be S -isomorphic if there exists an S -isomorphism between them. (In this case, we write $M \cong_S N$.)

Recall that we have adopted the convention that if M and M' are objects of \mathcal{M} , then $M \otimes M'$ is taken to mean the usual tensor product modulo \mathbb{Z} -torsion (and thus also lies in \mathcal{M}).

If we have an S -morphism

$$h: [A_1] \otimes M \rightarrow [A_2],$$

then h induces an S -morphism

$$\hat{h}: M \rightarrow \text{Hom}([A_1], [A_2]),$$

defined by $\hat{h}(m)(x) = h(m \otimes x)$ for $x \in [A_1], m \in M$. By Theorem 4.11, we can consider \hat{h} to be a map $M \rightarrow \text{Hom}(A_2, A_1)$. This gives us a map $g: \text{Hom}(A_1, A_2) \rightarrow \text{Hom}(M, \text{End}(A_1))$, defined by $g(\phi)(m) = \hat{h}(m) \circ \phi$, for $\phi: A_1 \rightarrow A_2$.

For an element ϕ of $\text{Hom}_{\mathcal{A}_S}(A_1, A_2)$, let $I_{\phi, S} \subset \text{End}(A_1) \otimes_{\mathbb{T}} \mathbb{T}_S$ be the ideal generated by the image of $M \otimes_{\mathbb{T}} \mathbb{T}_S$ under $g(\phi)$, and let I_{ϕ} be the ideal $I_{\phi, S} \cap \text{End}(A_1)$.

PROPOSITION 4.12: *The map g defined above is an S -isomorphism. In particular, g induces an isomorphism*

$$\text{Hom}_{\mathcal{A}_S}(A_1, A_2) \rightarrow \text{Hom}_{\mathcal{M}_S}(M, \text{End}(A_1)).$$

Moreover, if ϕ is an element of $\text{Hom}_{\mathcal{A}_S}(A_1, A_2)$, then $\ker_S \phi = A_1[I_{\phi}]$ “up to support on S .”

Proof: We construct an inverse to g in \mathcal{M}_S . We have a sequence of natural

S-morphisms

$$\begin{aligned}
\mathrm{Hom}(M, \mathrm{End}(A_1)) &\rightarrow \mathrm{Hom}(M, \mathrm{End}([A_1])) \\
&\rightarrow \mathrm{Hom}(M \otimes [A_1], \mathrm{End}([A_1]) \otimes [A_1]) \\
&\xrightarrow{h} \mathrm{Hom}([A_2], [A_1]) \\
&\rightarrow \mathrm{Hom}(A_1, A_2),
\end{aligned}$$

where the first and last morphisms come from Theorem 4.11, and the second sends a map f to the map $f \otimes 1$. It is straightforward to check that this provides an inverse to g .

Now fix a $\phi \in \mathrm{Hom}_{\mathcal{A}_S}(A_1, A_2)$, and an \mathfrak{m} outside S , and let I_ϕ be the corresponding ideal defined above. Since $[A_1]_{\mathfrak{m}} \otimes M_{\mathfrak{m}}$ maps onto $[A_2]_{\mathfrak{m}}$, the map $T_{\mathfrak{m}}A_1 \otimes M_{\mathfrak{m}} \rightarrow T_{\mathfrak{m}}A_2$ defined by sending $\beta \otimes m$ to $(\hat{h}(m))_{\mathfrak{m}}(\beta)$, is surjective. (The latter map can be obtained from the former by tensoring with $T_{\mathfrak{m}}J^{\min}$ and exploiting the isomorphism $T_{\mathfrak{m}}J' \cong T_{\mathfrak{m}}J^{\min} \otimes [J']$ for $J' = A_1$ or A_2 .)

Thus we can write any $\alpha \in T_{\mathfrak{m}}A_2$ as a sum of terms of the form

$$(\hat{h}(m_i))_{\mathfrak{m}}(\beta_i)$$

for some m_i in $M_{\mathfrak{m}}$ and β_i in $T_{\mathfrak{m}}A_1$. But by our construction of g , we have

$$\phi_{\mathfrak{m}}(\hat{h}(m_i))_{\mathfrak{m}}(\beta_i) = g(\phi)(m_i)\beta_i,$$

and the latter is in $I_\phi T_{\mathfrak{m}}A_1$ for all i . Thus $\phi_{\mathfrak{m}}$ maps α into $I_\phi T_{\mathfrak{m}}A_1$. Hence the image of $\phi_{\mathfrak{m}}$ is contained in $I_{\phi, \mathfrak{m}} T_{\mathfrak{m}}A_1$.

Conversely, observe that by construction, every element of I_ϕ factors through ϕ , since any element of I_ϕ is in the image of $g(\phi)$ and the definition of $g(\phi)$ involved composition with ϕ . Thus the image of $\phi_{\mathfrak{m}}$ contains $I_\phi T_{\mathfrak{m}}A_1$.

It follows that $\phi_{\mathfrak{m}} T_{\mathfrak{m}}A_2 = I_\phi T_{\mathfrak{m}}A_1$ for all \mathfrak{m} outside S . Since the former is the annihilator of $\ker \phi: A_1[\mathfrak{m}^\infty] \rightarrow A_2[\mathfrak{m}^\infty]$ and the latter is the annihilator of $A_1[\mathfrak{m}^\infty][I_\phi]$, we have $\ker_S \phi = A_1[I_\phi]$ up to support on S . ■

The behavior of certain invariants of J under isogeny can be studied using this formalism. In particular, let p be a prime, and suppose that J has purely toric reduction at p . Then the special fiber of the Néron model of J at p is an extension of a finite group scheme by a torus T , and we can consider the character group $X_p(J) = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(T, \mathbb{G}_m)$ of T . More generally, any object A of \mathcal{A} will also have purely toric reduction at p , and we can consider its character group $X_p(A)$.

LEMMA 4.13: *The character group $X_p(A)$ is a “rank one” \mathbb{T} -module.*

Proof: By construction, $X_p(A)$ has finite rank over \mathbb{Z} , and is torsion free as a \mathbb{Z} -module. It thus suffices to show that $X_p(A) \otimes \mathbb{Q}$ is free of rank one as a $\mathbb{T} \otimes \mathbb{Q}$ -module. Let \mathfrak{p} be a maximal ideal of $\mathbb{T} \otimes \mathbb{Q}$. It is then enough to show that $X_p(A)_{\mathfrak{p}}$ is free of rank one over $(\mathbb{T} \otimes \mathbb{Q})_{\mathfrak{p}}$.

Since $\mathbb{T} \otimes \mathbb{Q}$ is Gorenstein, $(\mathbb{T} \otimes \mathbb{Q})[\mathfrak{p}]$ is generated (as a $(\mathbb{T} \otimes \mathbb{Q})/\mathfrak{p}$ -vector space) by a single element σ . Since \mathbb{T} acts faithfully on J , it acts faithfully on A as well, and hence on $X_p(A)$. In particular there is an x in $X_p(A)_{\mathfrak{p}}$ not killed by σ . Let I be the annihilator of x in $\mathbb{T} \otimes \mathbb{Q}$. Then I is killed by some power of \mathfrak{p} , so $I[\mathfrak{p}]$ is nontrivial unless I is the zero ideal. But if $I[\mathfrak{p}]$ were nontrivial, then I would contain σ , which does not annihilate x by assumption. Thus I is the zero ideal, and $X_p(A)_{\mathfrak{p}}$ contains a free $(\mathbb{T} \otimes \mathbb{Q})_{\mathfrak{p}}$ -submodule for all \mathfrak{p} .

Counting \mathbb{Q} -ranks, we find that $X_p(A)_{\mathfrak{p}}$ is free of rank one over $(\mathbb{T} \otimes \mathbb{Q})_{\mathfrak{p}}$ for all \mathfrak{p} , since the rank of $X_p(A)$ is equal to the dimension of A , and hence to the \mathbb{Z} -rank of \mathbb{T} . ■

If A_1 and A_2 are two objects in \mathcal{A} , and $\phi: A_1 \rightarrow A_2$ is a \mathbb{T} -equivariant map defined over \mathbb{Q} , then ϕ induces a natural map $X_p(A_2) \rightarrow X_p(A_1)$. By abuse of notation, we denote this map by ϕ as well. (It will always be clear from the context whether we mean a morphism of abelian varieties or the map it induces on the character groups.)

PROPOSITION 4.14: *Let $A \in \mathcal{A}$. The natural map*

$$X_p(J^{\min}) \otimes \operatorname{Hom}(A, J^{\min}) \rightarrow X_p(A)$$

defined by $\phi \otimes x \mapsto \phi \circ x$ is an isomorphism in \mathcal{M}_S .

Proof: Let $T_l A$ denote the *covariant* l -adic Tate module of A , and $(T_l A)^t$ be the submodule of $T_l A$ defined in [SGA], Chapter 5. By Grothendieck’s orthogonality theorem, $(T_l A)^t$ is the orthogonal complement of a certain submodule $(T_l A^{\vee})^f$ of $T_l A^{\vee}$ under the Weil pairing; in particular, if $l^n x \in (T_l A)^t$ for some x in $T_l A$ and $n \in \mathbb{Z}$, then $x \in (T_l A)^t$ as well. It follows that the map $\operatorname{Hom}(T_l A, \mathbb{Z}_l(1)) \rightarrow \operatorname{Hom}((T_l A)^t, \mathbb{Z}_l(1))$ is surjective. Localizing at a maximal ideal \mathfrak{m} outside S , we obtain a surjection $T_{\mathfrak{m}} A \rightarrow \operatorname{Hom}((T_l A)^t, \mathbb{Z}_l(1))_{\mathfrak{m}}$, where as usual $T_{\mathfrak{m}} A$ denotes the \mathfrak{m} -adic contravariant Tate module. Since $(T_l A)^t \cong \operatorname{Hom}(X_p(A), \mathbb{Z}_l(1))$, by the results of [SGA], Chapter 5, this gives us a functorial surjection: $T_{\mathfrak{m}} A \rightarrow X_p(A)_{\mathfrak{m}}$. Similarly, we obtain a functorial surjection $T_{\mathfrak{m}} J^{\min} \rightarrow X_p(J^{\min})_{\mathfrak{m}}$.

These surjections fit into the commutative diagram:

$$\begin{array}{ccc} T_{\mathfrak{m}} J^{\min} \otimes \operatorname{Hom}(A, J^{\min})_{\mathfrak{m}} & \longrightarrow & T_{\mathfrak{m}} A \\ \downarrow & & \downarrow \\ X_p(J^{\min})_{\mathfrak{m}} \otimes \operatorname{Hom}(A, J^{\min})_{\mathfrak{m}} & \longrightarrow & X_p(A)_{\mathfrak{m}}, \end{array}$$

in which the horizontal maps are the natural ones, the right hand vertical map is the surjection defined above, and the left hand vertical map is the surjection defined above tensored with the identity. By Corollary 4.10, the upper horizontal map is surjective; since both vertical maps are surjections the lower horizontal map must be surjective as well. ■

We will also have use for the following result, which is an immediate consequence:

COROLLARY 4.15: *Let \mathfrak{m} be a maximal ideal outside S , and suppose that $X_p(J^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$. Then for any $A \in \mathcal{A}$, we have $T_{\mathfrak{m}} A \cong (X_p(A)_{\mathfrak{m}})^2$.*

Finally, we wish to interpret duality of abelian varieties in terms of the formalism we have constructed. The dual J^{\vee} of an abelian variety J in \mathcal{A} has a natural \mathbb{T} -action, in which $t \in \mathbb{T}$ acts on J^{\vee} by the dual of its action on J . Unfortunately, with this choice of action, J^{\vee} need not lie in \mathcal{A} . In fact, the abelian variety J^{\vee} will be a “twist” of some abelian variety in \mathcal{A} by the character χ we fixed at the beginning of this section, in a sense we will make precise below.

Let \mathcal{A}_{χ} denote the category of abelian varieties that are \mathbb{T} -equivariantly isogenous to J^{\vee} , and whose morphisms are \mathbb{T} -equivariant and defined over \mathbb{Q} . Duality of abelian varieties yields a functor $A \mapsto A^{\vee}$ from \mathcal{A} to \mathcal{A}_{χ} .

Consider also the category $\mathcal{A}_{1,\chi}$, whose objects are triples (A, A_{χ}, φ) , where $A \in \mathcal{A}$, $A_{\chi} \in \mathcal{A}_{\chi}$, and φ is a \mathbb{T} -equivariant isomorphism $A \rightarrow A_{\chi}$, defined over K , such that $\sigma\varphi = \chi(\sigma)\varphi\sigma$, for all $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$. A morphism in $\mathcal{A}_{1,\chi}$ from (A, A_{χ}, φ) to $(A', A'_{\chi}, \varphi')$ is a pair of morphisms $A \rightarrow A'$ (in \mathcal{A}) and $A_{\chi} \rightarrow A'_{\chi}$ (in \mathcal{A}_{χ}) that makes the obvious diagram commute. There are natural forgetful functors $\mathcal{A}_{1,\chi} \rightarrow \mathcal{A}$ and $\mathcal{A}_{1,\chi} \rightarrow \mathcal{A}_{\chi}$.

LEMMA 4.16: *The functors $\mathcal{A}_{1,\chi} \rightarrow \mathcal{A}$ and $\mathcal{A}_{1,\chi} \rightarrow \mathcal{A}_{\chi}$ are equivalences of categories.*

Proof: Both functors are clearly fully faithful; it suffices to show that they are essentially surjective. Fix an $A \in \mathcal{A}$; then it is enough to show that we have an $A' \in \mathcal{A}_{\chi}$ and a map $\varphi: A \rightarrow A'$ as above.

Fix an isogeny $\phi: J \rightarrow A$. Multiplying by a sufficiently large integer n , we may assume that $\ker \phi$ contains the kernel of the map $\psi: J \rightarrow J^\vee$. Let $A' = J^\vee / \psi(\ker \phi)$. The natural map $J \rightarrow A'$ induced by ψ has the same kernel as ϕ , and thus descends to an isomorphism φ from A to A' . Then (A, A', φ) is an object of $\mathcal{A}_{1,\chi}$ mapping to A . The proof for the functor $\mathcal{A}_{1,\chi} \rightarrow \mathcal{A}_\chi$ is similar. ■

The upshot is that \mathcal{A} and \mathcal{A}_χ are equivalent categories. In particular the above equivalences induce a functor $A \mapsto A_\chi$ from \mathcal{A} to \mathcal{A}_χ and the inverse functor $A' \mapsto A'_{\chi^{-1}}$, and these functors are equivalences of categories. (These functors can be thought of as “twisting” an abelian variety A by the character χ or its inverse. In particular A and A_χ become isomorphic over the splitting field of χ , via an isomorphism φ that satisfies $\sigma\varphi = \chi(\sigma)\varphi\sigma$ for all σ in $\text{Gal}(K/\mathbb{Q})$.) This allows us to define a natural duality operation on \mathcal{A} , by taking $A^\dagger = (A^\vee)_{\chi^{-1}}$.

This operation interacts with our equivalence of categories $\mathcal{A}_S \rightarrow \mathcal{M}_S$ as follows:

PROPOSITION 4.17: *The \mathbb{T}_S -module $[(J^{\min})^\dagger]^*$ is locally free of rank one. If L denotes its inverse as a \mathbb{T}_S -module, then we have $[A^\dagger] \cong [A]^* \otimes L$ for any $A \in \mathcal{A}$.*

Proof: The Weil pairing induces an isomorphism,

$$T_{\mathfrak{m}}(J^{\min})^\vee \cong \text{Hom}(T_{\mathfrak{m}}J^{\min}, \mathbb{Z}_l(1)).$$

Since $T_{\mathfrak{m}}(J^{\min})^\vee$ and $T_{\mathfrak{m}}(J^{\min})^\dagger$ are isomorphic as $\mathbb{T}_{\mathfrak{m}}$ -modules (but not Galois modules), $T_{\mathfrak{m}}(J^{\min})^\dagger$ is isomorphic to $\text{Hom}(T_{\mathfrak{m}}J^{\min}, \mathbb{Z}_l(1))$. As $T_{\mathfrak{m}}J^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$, it follows that $(T_{\mathfrak{m}}(J^{\min})^\dagger)^*$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. By Corollary 4.10, $[(J^{\min})^\dagger]^*$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$.

Now for any $A \in \mathcal{A}$, we have

$$[A^\dagger] \cong_S \text{Hom}(A^\dagger, J^{\min}) \cong_S \text{Hom}((J^{\min})^\dagger, A) \cong_S \text{Hom}([A], [(J^{\min})^\dagger]).$$

We have $[(J^{\min})^\dagger] \cong \mathbb{T}_S^* \otimes L$, so $\text{Hom}([A], [(J^{\min})^\dagger]) \cong_S [A]^* \otimes L$ by Lemma 2.5. ■

5. The character group of the pq -new subvariety

We now return to the situation considered in the introduction. Fix a squarefree integer N , with an even number of prime divisors, and let Γ be a congruence subgroup of level M for some integer prime to N . For R a divisor of N , and D a divisor of R that is divisible by an even number of primes, we let $J^{D,R}$ denote

the abelian variety $J^D(\Gamma_0(N/D) \cap \Gamma)_{R/D-\text{new}}$. Let \mathbb{T}^R denote the subalgebra of $\text{End}(J^{1,R})$ generated by the Hecke operators T_l for l prime, (when l divides MN we take T_l to be defined as in [Ril]), and the diamond bracket operators, and let \mathcal{A}^R denote the category of abelian varieties defined over \mathbb{Q} that are \mathbb{T}^R -equivariantly isogenous to $J^{1,R}$. Then for each D as above, $J^{D,R}$ is an object of \mathcal{A}^R . Since we will be primarily interested in \mathcal{A}^N , we let $J^D = J^{D,N}$ for any divisor D of N that is divisible by an even number of primes. Similarly, we take $\mathcal{A} = \mathcal{A}^N$, $\mathbb{T} = \mathbb{T}^N$, and so forth. Let S^R be the set of maximal ideals of \mathbb{T}^R that are either Eisenstein (i.e., whose associated mod l Galois representation is *reducible*), or of residue characteristic 2 or 3, and take $S = S^N$.

As mentioned in Remark 4.1, the hypotheses (and hence the results) of the previous section hold for \mathcal{A}^R . We fix, once and for all, a J_1^{\min} inside \mathcal{A}^1 such that $T_{\mathfrak{m}}J_1^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}^1$ for all non-Eisenstein \mathfrak{m} . If we then take $J_R^{\min} = (J_1^{\min})_{R-\text{new}}$, then $T_{\mathfrak{m}}J_R^{\min}$ is free of rank two over \mathbb{T}^R for all non-Eisenstein maximal ideals \mathfrak{m} of \mathbb{T}^R . We can thus consider the functors $J \rightarrow [J]_R = \text{Hom}(J, J_R^{\min})$ for $J \in \mathcal{A}^R$, and use them to study isogenies between the abelian varieties $J^{D,R}$ for various D . By abuse of notation, we often denote $[J]_R$ simply by $[J]$ if it is clear from the context which category J belongs to.

The abelian variety $J^{1,R}$ (and hence every abelian variety in \mathcal{A}^R) has purely toric reduction at any prime p dividing R (see for instance the proof of Theorem 3.10 in [Ril].) Thus we can consider the character groups of $J^{D,R}$ at such primes. We then have relationships between these character groups and the module $[J^{D,R}]$, given by Proposition 4.14. Thus, if we can compute the character groups $X_p(J^{D,R})$, we can hope to obtain information about maps between varieties of this form from this data. We carry out this program in the remainder of this paper.

Let R' be an integer dividing N and divisible by R . By [CS], Theorem 8.2, the injection $J^{D,R'} \rightarrow J^{D,R}$ induces an injection on the maximal tori of the special fibers at p , and hence a surjection $X_p(J^{D,R}) \rightarrow X_p(J^{D,R'})$. Moreover, both $X_p(J^{D,R})$ and $X_p(J^{D,R'})$ are free \mathbb{Z} -modules, and there are isomorphisms:

$$X_p(J^{D,R}) \otimes \mathbb{Q} \cong \mathbb{T}^R \otimes \mathbb{Q} \quad \text{and} \quad X_p(J^{D,R'}) \otimes \mathbb{Q} \cong \mathbb{T}^{R'} \otimes \mathbb{Q}.$$

This actually suffices to determine the isomorphism class of $X_p(J^{D,R'})$ in terms of $X_p(J^{D,R})$.

More generally, let A be any finite flat \mathbb{Z} -algebra, and suppose that $A \otimes \mathbb{Q}$ is Gorenstein. (This is easily seen to hold for $A = \mathbb{T}^R$ by results of the previous section.) Let A' be a quotient of A that is flat over \mathbb{Z} . Let X be a finitely

generated A -module such that $X \otimes \mathbb{Q}$ is free of rank one over $A \otimes \mathbb{Q}$, and let Y be a quotient of X that is a faithful A' -module, and flat over \mathbb{Z} .

LEMMA 5.1: *Suppose X and Y are as above, and let L be the kernel of the map $X \rightarrow Y$. For an ideal I of A , let I^\perp be the annihilator of I in A . (This notation is due to Emerton [Em].) Let K be the kernel of the map $A \rightarrow A'$. Then $L = X[K^\perp]$, and we have $Y \cong X \otimes_A A'$ (recall our convention, in which we take the tensor product to be the usual tensor product modulo \mathbb{Z} -torsion).*

Proof: We begin by showing that K has finite index in $K^{\perp\perp}$. (Note that $K^{\perp\perp}$ trivially contains K .) Since $\hat{A} = A \otimes \mathbb{Q}$ is a zero-dimensional Gorenstein ring, we fix an isomorphism $\hat{A} \cong \text{Hom}_{\mathbb{Q}}(\hat{A}, \mathbb{Q})$. For any ideal I of \hat{A} , this isomorphism identifies I^\perp with $\text{Hom}_{\mathbb{Q}}(\hat{A}/I, \mathbb{Q})$. In particular $\dim_{\mathbb{Q}} I^\perp = \dim_{\mathbb{Q}} \hat{A} - \dim_{\mathbb{Q}} I$, and so $\dim_{\mathbb{Q}} I^{\perp\perp} = \dim_{\mathbb{Q}} I$. Since I is contained in $I^{\perp\perp}$, the two must be equal. It follows that K is contained in $K^{\perp\perp}$ with finite index, as required.

We have an exact sequence:

$$0 \rightarrow L \rightarrow X \rightarrow Y \rightarrow 0.$$

Since Y is an A' -module, K annihilates Y , so $KX \subset L$. Now $KX \subset X[K^\perp]$, by definition of K^\perp . Moreover, since $X \otimes \mathbb{Q}$ is free over \hat{A} ,

$$X[K^\perp] \otimes \mathbb{Q} = \hat{A}[K^\perp] = K^{\perp\perp} X \otimes \mathbb{Q}.$$

Thus $X[K^\perp]$ contains $K^{\perp\perp} X$ with finite index, and hence contains KX with finite index. Let n be the index of KX in $X[K^\perp]$. Suppose x in $X[K^\perp]$ maps to a nonzero element of Y . Then nx maps to the zero element of Y , so (since we assumed Y is flat over \mathbb{Z}), x maps to 0 in Y . Thus $X[K^\perp] \subset L$.

Since $X \otimes \mathbb{Q}$ is generated by a single element over $A \otimes \mathbb{Q}$, $Y \otimes \mathbb{Q}$ is generated by a single element over $A' \otimes \mathbb{Q}$; since Y is faithful over A' , $Y \otimes \mathbb{Q}$ is free of rank one over $A' \otimes \mathbb{Q}$. We thus have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & X & \longrightarrow & Y \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L \otimes \mathbb{Q} & \longrightarrow & A \otimes \mathbb{Q} & \longrightarrow & A' \otimes \mathbb{Q} \longrightarrow 0. \end{array}$$

Since the kernel of the map $A \otimes \mathbb{Q} \rightarrow A' \otimes \mathbb{Q}$ is killed by K^\perp , so is $L \otimes \mathbb{Q}$. Hence L is killed by K^\perp , so $L = X[K^\perp]$. In particular L contains KX with finite index. Hence $Y \cong X/L$ is isomorphic to X/KX , modulo \mathbb{Z} -torsion in X/KX . The latter is precisely $X \otimes_A A'$. ■

We record the following useful corollary before we proceed:

COROLLARY 5.3: *Let X, Y, A, A' and K be as above. Then $Y^* \cong X^*[K]$. In particular, $X^*[K] \cong (X \otimes_A A')^*$ for any X and K as above.*

Proof: We have an exact sequence

$$0 \rightarrow L \rightarrow X \rightarrow Y \rightarrow 0.$$

Taking duals gives us the sequence:

$$0 \rightarrow Y^* \rightarrow X^* \rightarrow L^* \rightarrow 0.$$

The result now follows by the previous lemma. \blacksquare

The upshot of Lemma 5.1 is the following corollary:

COROLLARY 5.3: *Let R and R' be divisors of N , and suppose R divides R' . For any p dividing R , we have $X_p(J^{D,R'}) \cong X_p(J^{D,R}) \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

We now fix two distinct primes p and q dividing N . Let D be a divisor of N , divisible by an even number of primes but *not* divisible by p or q . The previous result, together with the following result of Ribet, establishes a relationship (Proposition 5.7, below) between the character groups $X_p(J^{D,Dpq})$ and $X_q(J^{Dpq,Dpq})$.

THEOREM 5.4: *There is a \mathbb{T}^D -equivariant exact sequence:*

$$0 \rightarrow X_q(J^{Dpq,Dpq}) \rightarrow X_p(J^{D,D}) \rightarrow X_p(J^D(\Gamma_0(N/Dq) \cap \Gamma))^2 \rightarrow 0.$$

Proof: When $D = 1$, this is [Ri1], Theorem 4.1. The same argument extends to the case where $D > 1$ using Buzzard's analogue of the Deligne–Rapoport theorem [Bu1], except that Buzzard's result assumes that the level structure is contained in $\Gamma_1(r)$ for some integer $r > 3$. An argument that removes this restriction can be found in the appendix. \blacksquare

Dualizing, and applying Lemma 5.1, we find that

$$X_q(J^{Dpq,Dpq})^* \cong X_p(J^{D,D})^* \otimes_{\mathbb{T}^D} \mathbb{T}^{Dpq}.$$

This is almost what we need; it remains to compare $X_p(J^{D,D})^*$ with $X_p(J^{D,D})$. We do this via the monodromy pairing on $X_p(J^{D,D})$. This is a bilinear pairing:

$$X_p(J^{D,D}) \times X_p((J^{D,D})^\vee) \rightarrow \mathbb{Z},$$

for which the adjoint of T_l acting on $J^{D,D}$ is T_l^\vee acting on $(J^{D,D})^\vee$.

LEMMA 5.5: Let $\Gamma' = \Gamma_0(N/D) \cap \Gamma$, and $R = NM/D$ be the level of Γ' . There exists a Hecke-equivariant isomorphism $\psi: J^{D,D} \rightarrow (J^{D,D})^\vee$, defined over $\mathbb{Q}(\mu_R)$. Moreover, for any $\sigma \in \text{Gal}(\mathbb{Q}(\mu_R)/\mathbb{Q})$, $\sigma\psi = \chi(\sigma)\psi\sigma$, where $\chi: \text{Gal}(\mathbb{Q}(\mu_R)/\mathbb{Q}) \rightarrow \mathbb{T}^\times$ is the character such that if $\sigma\zeta_R = \zeta_R^d$ for a primitive root of unity ζ_R , then $\chi(\sigma) = \langle d \rangle$.

Proof: Let w_R be the Atkin–Lehner involution on $\Gamma' = \Gamma_0(N/D) \cap \Gamma$; that is, the involution of $J^{D,D} = J^D(\Gamma')$ corresponding to the double coset $\Gamma' \begin{pmatrix} 0 & 1 \\ R & 0 \end{pmatrix} \Gamma'$. The discussion on page 193 of [Sh], together with [Sh], Proposition 3.54, shows that the Rosati involution induced by the canonical polarization θ of $J^D(\Gamma')$ sends an element T_l of \mathbb{T} to $w_R T_l w_R$, for any l prime to D . (Strictly speaking, Shimura only covers the case in which $D = 1$ and l is arbitrary, but one can just as easily work locally at l for any l prime to D and his argument carries over without change.)

For l dividing D , one can view $X^D(\Gamma')$ as a moduli space of abelian surfaces with quaternionic multiplication by the quaternion algebra B of discriminant D , with associated level structure. (See [Bu1] for details.) From this perspective, T_l is induced by the map on $X^D(\Gamma')$ that sends an abelian surface A to $A/A[\mathfrak{l}]$, where \mathfrak{l} is the unique two-sided maximal ideal of index l in a maximal order of B_D . Since this map is an isomorphism, the Rosati involution sends T_l to its inverse. But it is easy to see from the moduli definition of T_l that $T_l^2 = \langle l \rangle$. Thus the Rosati involution sends T_l to $\langle l \rangle^{-1} T_l$. Moreover, by [Sh], Proposition 3.55, we have $w_R T_l w_R = \langle l \rangle^{-1} T_l$ on $J^1(\Gamma_0(D) \cap \Gamma')$, and hence on $J^D(\Gamma)$ as well by Jacquet–Langlands.

Finally, it is easy to verify that both the Rosati involution and conjugation by w_R send $\langle d \rangle$ to $\langle d \rangle^{-1}$ for any d . (For the former, simply note that $\langle d \rangle$ is induced by an automorphism of $X^D(\Gamma')$; the latter is part of [Sh], Proposition 3.55.) Since the elements T_l and $\langle d \rangle$ as l and d vary generate \mathbb{T} , we find that the Rosati involution is simply conjugation by w_R . It is then easy to check that $\psi = w_R \theta$ is Hecke-equivariant. The second statement (and the fact that ψ is defined over $\mathbb{Q}(\mu_R)$), follow from the identity $\sigma w_R = \chi(\sigma) w_R \sigma$ ([Sh], Proposition 7.8). ■

The map ψ defined above induces a duality operation $A \rightarrow A^\dagger$, by results of the previous section. It also defines an isomorphism $J^{D,D} \cong (J^{D,D})^\vee$ over $\mathbb{Q}(\mu_R)$. Since the character groups at p of $(J^{D,D})^\vee$ are the same whether we consider the varieties over \mathbb{Q} or over $\mathbb{Q}(\mu_R)$, the map ψ induces an isomorphism $X_p(J^{D,D})^\vee \cong X_p(J^{D,D})$ (as \mathbb{T}^D -modules but not Galois modules). Since ψ is \mathbb{T}^D -equivariant, in the sense that $T_l^\vee \psi = \psi T_l$, this identification allows us to

view the monodromy pairing

$$X_p(J^{D,D}) \times X_p((J^{D,D})^\vee) \rightarrow \mathbb{Z}$$

as a \mathbb{T}^D -equivariant bilinear pairing on $X_p(J^{D,D})$, i.e., as a \mathbb{T}^D -equivariant map $X_p(J^{D,D}) \rightarrow X_p(J^{D,D})^*$. The cokernel of this map is naturally isomorphic to the component group of $J^{D,D}$ at p (c.f. [Ri1], Theorem 2.3), and Ribet has shown:

PROPOSITION 5.6: *The component group of $J^{D,D}$ at p is supported on S^D .*

Proof: Since p is prime to D , the methods of 5.7, which cover the discriminant 1 case, carry over to the case of arbitrary discriminant once one invokes Buzzard's analogue of the Deligne–Rapoport theorem ([Bul]), and works with abelian surfaces with quaternionic multiplication rather than elliptic curves. As before, see the appendix for an extension of Buzzard's result to the case in which the level structure does not contain $\Gamma_1(r)$ for some $r > 4$. ■

With this in hand, we can now show:

PROPOSITION 5.7: *We have $X_p(J^{D,Dpq}) \cong_{S^{Dpq}} X_q(J^{Dpq,Dpq})^*$.*

Proof: Corollary 5.3 identifies $X_p(J^{D,Dpq})$ with $X_p(J^{D,D}) \otimes_{\mathbb{T}^D} \mathbb{T}^{Dpq}$. On the other hand, the \mathbb{Z} -dual of the exact sequence of Theorem 5.4 identifies $X_q(J^{Dpq,Dpq})^*$ with the module $X_p(J^{D,D})^* \otimes_{\mathbb{T}^D} \mathbb{T}^{Dpq}$ by Lemma 5.1. Since $X_p(J^{D,D}) \cong_{S^D} X_p(J^{D,D})^*$ by the above discussion, it follows that

$$X_p(J^{D,Dpq}) \cong_{S^{Dpq}} X_q(J^{Dpq,Dpq})^*. \quad \blacksquare$$

We will also need of a compatibility property that the above isomorphisms possess. Let p and p' be two distinct primes dividing D , take $D' = D/(pp')$, and choose distinct primes q_1 and q_2 dividing N but not D . Let $\underline{q} = q_1 q_2$.

LEMMA 5.8: *Let $f: J^{D'\underline{q},D'\underline{q}} \rightarrow J^{D',D'}$ and $g: J^{D\underline{q},D\underline{q}} \rightarrow J^{D,D}$ be maps such that there is a commutative diagram:*

$$\begin{array}{ccc} X_p(J^{D\underline{q},D\underline{q}}) & \longrightarrow & X_{p'}(J^{D'\underline{q},D'\underline{q}}) \\ g^\dagger \downarrow & & \downarrow f^\dagger \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D'}) \end{array}$$

in which the horizontal maps are those of Theorem 5.4. Let \bar{f} denote the restriction of f to the pp' -new subvariety $J^{D'q,Dq}$ of $J^{D'q,D'q}$. Then we have a commutative diagram:

$$\begin{array}{ccc} X_p(J^{Dq,Dq}) & \longrightarrow & X_{p'}(J^{D'q,Dq})^* \\ g^\dagger \downarrow & & \downarrow \bar{f}^* \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D})^*, \end{array}$$

in which the horizontal maps are the isomorphisms constructed in Proposition 5.7.

Proof: Consider the diagram:

$$\begin{array}{ccccccc} X_p(J^{Dq,Dq}) & \longrightarrow & X_{p'}(J^{D'q,D'q}) & \longrightarrow & X_{p'}(J^{D'q,D'q})^* & \longleftarrow & X_{p'}(J^{D'q,Dq})^* \\ g^\dagger \downarrow & & f^\dagger \downarrow & & \downarrow f^* & & \downarrow \bar{f}^* \\ X_p(J^{D,D}) & \longrightarrow & X_{p'}(J^{D',D'}) & \longrightarrow & X_{p'}(J^{D',D'})^* & \longleftarrow & X_{p'}(J^{D',D})^*. \end{array}$$

Here the left-hand square is that appearing in the hypothesis of the lemma. The middle square commutes because f^\dagger and f are adjoints under the monodromy pairing. The right-hand square is the \mathbb{Z} -dual of the diagram:

$$\begin{array}{ccc} X_{p'}(J^{D',D}) & \longrightarrow & X_{p'}(J^{D',D'}) \\ f \downarrow & & \downarrow \bar{f} \\ X_{p'}(J^{D'q,Dq}) & \longrightarrow & X_{p'}(J^{D'q,D'q}), \end{array}$$

which clearly commutes since \bar{f} is the restriction of f to the pp' -new subvariety, and the horizontal maps are just inclusions of the pp' -new subvarieties into the corresponding variety.

The isomorphisms:

$$\begin{aligned} X_p(J^{D,D}) &\rightarrow X_{p'}(J^{D',D})^*, \\ X_p(J^{Dq,Dq}) &\rightarrow X_{p'}(J^{D'q,Dq})^* \end{aligned}$$

are obtained as the compositions of the two leftmost horizontal map with the “inverse” of the rightmost horizontal map (bearing in mind that the image of the rightmost horizontal map is the same as the image of the composition of the two rightmost), in the corresponding row of the above diagram. The commutativity of the desired square thus follows immediately. ■

We will also need the following result about character groups in Section 7.

PROPOSITION 5.9: *Let q_1 and q_2 be two distinct primes dividing N but not D . Let $Y_{q_1} = X_{q_1}(J^{D_{q_1}q_2, D_{q_1}q_2})$ and let $X_{q_2} = X_{q_2}(J^D(\Gamma_0(N/Dq_1) \cap \Gamma))^2$. Then there is a natural S^D -isomorphism of \mathbb{T}^D -modules:*

$$Y_{q_1}^*/Y_{q_1} \cong_{S^D} X_{q_2}/(T_{q_1}^2 - \langle q_1 \rangle)X_{q_2}.$$

(Here Y_{q_1} is considered as a submodule of $Y_{q_1}^*$ via the monodromy pairing.)

Proof: In [Ri1], Ribet constructs an exact sequence:

$$0 \rightarrow K \rightarrow X_{q_2}/(T_{q_1}^2 - \langle q_1 \rangle)X_{q_2} \rightarrow Y_{q_1}^*/Y_{q_1} \rightarrow C \rightarrow 0,$$

where K and C are supported on S^D , in the that case $D = 1$ and $\Gamma = \Gamma_0(N)$. As with Theorem 5.4, its arguments carry over to our setting once one invokes the analogue of the Deligne–Rapoport theorem found in the appendix. ■

6. Induced maps on character groups

In light of the character group computations of the previous section, we might hope to study morphisms in \mathcal{A}^N by studying the maps they induce on character groups. The main tool we will use to do this is the following result, whose proof will occupy this section as well as the next.

THEOREM 6.1: *Fix D dividing N . For any prime p dividing D , the natural map*

$$[J^{D,D}] \rightarrow \text{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))$$

that sends a map $J_D^{\min} \rightarrow J^{D,D}$ to the induced map on character groups at p is an S^D -isomorphism.

We will prove this theorem by constructing isomorphisms:

$$[J^{D,D}]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$$

for each \mathfrak{m} outside S^D . Although the isomorphisms we construct in this way are in some sense natural, showing any sort of compatibility between any two of them seems difficult. The following lemma provides us a way around this difficulty:

LEMMA 6.2: *Let M and N be $\mathbb{T}_{\mathfrak{m}}^D$ -modules, flat over \mathbb{Z} , and suppose that $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$ are free of rank one over $\mathbb{T}_{\mathfrak{m}}^D \otimes \mathbb{Q}$. Suppose we have a $\mathbb{T}_{\mathfrak{m}}^D$ module H , with no \mathbb{Z} -torsion, and an isomorphism $f: M \otimes H \rightarrow N$. Then the map*

$M \otimes \operatorname{Hom}(M, N) \rightarrow N$ given by evaluation is an isomorphism. Moreover, there is a natural map $g: H \rightarrow \operatorname{Hom}(M, N)$ such that the diagram

$$\begin{array}{ccc} M \otimes H & \xrightarrow{f} & N \\ 1 \otimes g \downarrow & & \downarrow \operatorname{Id} \\ M \otimes \operatorname{Hom}(M, N) & \longrightarrow & N \end{array}$$

commutes, where the right-hand vertical map is the identity. Finally, if $H \cong \operatorname{Hom}(M, N)$, then g is an isomorphism.

Proof: The natural map g takes an element h of H to the map $m \mapsto f(h \otimes m)$. The commutativity of the above diagram is trivial. Moreover, the upper horizontal map is an isomorphism by hypothesis, and the right-hand vertical map is the identity. It follows that the lower horizontal map is surjective, and hence (by counting ranks over \mathbb{Z}) an isomorphism. (Recall that $M \otimes \operatorname{Hom}(M, N)$ is torsion free by convention.)

It remains to prove that if $H \cong \operatorname{Hom}(M, N)$, then the natural map given above is an isomorphism. Composing the inverse of the isomorphism $H \rightarrow \operatorname{Hom}(M, N)$ with g , we obtain an endomorphism σ of $\operatorname{Hom}(M, N)$, such that $g(H) \subset \operatorname{Hom}(M, N)$ is equal to $\sigma \operatorname{Hom}(M, N)$. It thus suffices to show that σ is a unit in $\operatorname{End}(\operatorname{Hom}(M, N))$. The map $M \otimes \operatorname{Hom}(M, N) \cong N$ induces an inclusion of $\operatorname{End}(\operatorname{Hom}(M, N))$ into $\operatorname{End}(N)$. It follows that the image of $M \otimes H$ in N , under the composition of the left-hand vertical map and the bottom horizontal map, is equal to σN . By the commutativity of the diagram, σ is a unit in $\operatorname{End}(N)$. Suppose σ were not a unit in $\operatorname{End}(\operatorname{Hom}(M, N))$. Then there would be a maximal ideal \mathfrak{m}_1 of $\operatorname{End}(\operatorname{Hom}(M, N))$ containing σ . Since $\operatorname{End}(\operatorname{Hom}(M, N))$ is contained in $\operatorname{End}(N)$ with a finite index, there is a maximal ideal $\hat{\mathfrak{m}}_1$ of $\operatorname{End}(N)$ containing \mathfrak{m}_1 . In particular, $\hat{\mathfrak{m}}_1$ would contain σ , and so σ would not be a unit in $\operatorname{End}(N)$. This is a contradiction, so σ is a unit in $\operatorname{End}(\operatorname{Hom}(M, N))$, as required. ■

The usefulness of this lemma can be seen in the following corollary:

COROLLARY 6.3: *Let $J \in \mathcal{A}^D$, p be a prime dividing R , \mathfrak{m} a maximal ideal of \mathbb{T}^R outside S_D , and suppose that $[J]_{\mathfrak{m}} \cong \operatorname{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$. Then the natural map $[J]_{\mathfrak{m}} \rightarrow \operatorname{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$ is an isomorphism.*

Proof: By Proposition 4.14, we have a natural isomorphism:

$$X_p(J_D^{\min})_{\mathfrak{m}} \otimes [J]_{\mathfrak{m}} \rightarrow X_p(J)_{\mathfrak{m}}.$$

By Lemma 6.2, this induces a map $[J]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$, which is easily seen to be the natural map $[J]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min}), X_p(J))_{\mathfrak{m}}$. Since the two modules are isomorphic, this map is an isomorphism by Lemma 6.2. \blacksquare

In particular, in order to prove Theorem 6.1, it suffices to construct, for each \mathfrak{m} outside S^D , an arbitrary isomorphism

$$[J^{D,D}]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}.$$

We begin by doing this for maximal ideals \mathfrak{m} with a certain property.

Definition 6.4: Suppose that D divides N , and is divisible by an even number of primes, and let \mathfrak{m} be a maximal ideal of \mathbb{T}^D such that the representation $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then \mathfrak{m} is **controllable** at p (for some p dividing N) if one of the following conditions holds:

1. $\bar{\rho}_{\mathfrak{m}}$ is not finite at p , or
2. $\bar{\rho}_{\mathfrak{m}}$ is unramified at p , $p \neq l$, and $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_p)$ is not a scalar, or
3. $p = l$, and $l \neq 2$, or
4. $p = l = 2$, and the restriction of $\bar{\rho}_{\mathfrak{m}}$ to a decomposition group at 2 is not contained in the scalar matrices.

The point of this definition is the following

LEMMA 6.5: *Suppose that \mathfrak{m} is controllable at p . Then $X_p(J_D^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}^D$.*

Proof: By Nakayama's lemma, and the fact that $X_p(J_D^{\min})_{\mathfrak{m}}$ is faithful over $\mathbb{T}_{\mathfrak{m}}^D$, it suffices to show that the quotient $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension one. We will prove the lemma in this form.

Recall that in the proof of Proposition 4.14 we constructed a surjection:

$$T_{\mathfrak{m}}J_D^{\min} \rightarrow X_p(J_D^{\min})_{\mathfrak{m}}.$$

Tensoring over $\mathbb{T}_{\mathfrak{m}}^D$ with $\mathbb{T}^D/\mathfrak{m}$, we obtain a surjection

$$J_D^{\min}[\mathfrak{m}]^{\vee} \rightarrow X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min}).$$

By our construction of J_D^{\min} , $J_D^{\min}[\mathfrak{m}]^{\vee}$ is a single copy of $\bar{\rho}_{\mathfrak{m}}^{\vee}$, as a Galois-module. In particular it is two-dimensional over $\mathbb{T}^D/\mathfrak{m}$. On the other hand, $X_p(J_D^{\min})$ is a faithful \mathbb{T}^D -module, so $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ has dimension at least one. If its dimension is exactly one we are done.

Suppose its dimension is greater than one. Then the above surjection:

$$J_D^{\min}[\mathfrak{m}]^\vee \rightarrow X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$$

is an isomorphism, respecting the action of a decomposition group of $G_{\mathbb{Q}}$ above p . (Here inertia acts trivially on $X_p(J_D^{\min})$.) In particular, $\bar{\rho}_{\mathfrak{m}}^\vee$ (and hence $\bar{\rho}_{\mathfrak{m}}$) is finite at p .

Suppose first that $p \neq l$. Then Frob_p acts on $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ via the Atkin–Lehner operator $\langle p \rangle^{-1}w_p$, as $X_p(J_D^{\min})$ can be identified with a subspace of $X_p(J^{1,D})$, on which this is well-known. Since w_p acts as the scalar $-T_p$ on this space, it follows that $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_p)$ is a scalar. In particular, \mathfrak{m} cannot be controllable at p in this case.

Suppose finally that $p = l$. Since the determinant of $\bar{\rho}_{\mathfrak{m}}^\vee$ is the mod l cyclotomic character times an even character of finite order, $\bar{\rho}_{\mathfrak{m}}^\vee$ is ramified at p unless $p = l = 2$. Thus we cannot have an isomorphism

$$J_D^{\min}[\mathfrak{m}]^\vee \rightarrow X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$$

unless $p = l = 2$. If $p = l = 2$, then we find as before that the action of Frob_p on $X_p(J_D^{\min})/\mathfrak{m}X_p(J_D^{\min})$ is by scalars, and hence \mathfrak{m} again cannot be controllable at p . ■

When this criterion is satisfied for some prime dividing D , we may easily construct the desired isomorphism;

COROLLARY 6.6: *Let p divide D , and suppose \mathfrak{m} is not Eisenstein and controllable at at least one prime dividing D . Then $[J^{D,D}]_{\mathfrak{m}}$ is isomorphic to $\text{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}$ for any \mathfrak{m} outside S^D .*

Proof: If \mathfrak{m} is controllable at p , then $X_p(J_D^{\min})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$. On the other hand, by Proposition 4.14, we have

$$X_p(J^{D,D})_{\mathfrak{m}} \cong [J^{D,D}]_{\mathfrak{m}} \otimes X_p(J_D^{\min})_{\mathfrak{m}}.$$

The result follows immediately.

If \mathfrak{m} is not controllable at p , there is another prime q dividing D at which \mathfrak{m}

is controllable. Then we have a sequence of isomorphisms:

$$\begin{aligned}
 \operatorname{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))_{\mathfrak{m}} & \\
 &\cong \operatorname{Hom}(X_p(J_D^{\min}), X_q(J^{\frac{D}{pq},D})^*)_{\mathfrak{m}} \quad (\text{Proposition 5.7}) \\
 &\cong (X_p(J_D^{\min}) \otimes X_q(J^{\frac{D}{pq},D}))_{\mathfrak{m}}^* \quad (\text{Lemma 2.5}) \\
 &\cong (X_p(J_D^{\min}) \otimes [J^{\frac{D}{pq},D}])_{\mathfrak{m}}^* \quad (\text{controllability at } q) \\
 &\cong X_p(J^{\frac{D}{pq},D})_{\mathfrak{m}}^* \quad (\text{Proposition 4.14}) \\
 &\cong X_q(J^{D,D})_{\mathfrak{m}} \quad (\text{Proposition 5.7}) \\
 &\cong [J^{D,D}]_{\mathfrak{m}} \quad (\text{controllability at } q). \quad \blacksquare
 \end{aligned}$$

It remains to deal with those maximal ideals \mathfrak{m} that are not controllable at *any* prime dividing D . We do so in the next section, by “raising the level”. Specifically, we introduce two auxiliary primes q_1 and q_2 , prime to NM , such that \mathfrak{m} lifts to a maximal ideal of $\tilde{\mathfrak{m}}$ of $\mathbb{T}^D(\Gamma_0(q_1 q_2 N/D) \cap \Gamma)$ that is *new* at q_1 and q_2 and *controllable* at these primes. Then $\tilde{\mathfrak{m}}$ descends to a maximal ideal of $\mathbb{T}^{Dq_1 q_2}(\Gamma_0(N/D) \cap \Gamma)$ (because $\tilde{\mathfrak{m}}$ is new) and an analogue of Corollary 6.6 holds for the abelian variety $J^{Dq_1 q_2}(\Gamma_0(N/D) \cap \Gamma)$ at $\tilde{\mathfrak{m}}$, as $\tilde{\mathfrak{m}}$ is controllable at q_1 . We then exploit a geometric relationship between this abelian variety and $J^{D,D}$ to prove the desired result for $J^{D,D}$.

7. Raising the level

The key to the level raising argument is the following result:

LEMMA 7.1: *Let D be the product of an even number of distinct primes, and Γ a congruence subgroup of level M prime to D . Let \mathfrak{m} be a non-Eisenstein maximal ideal of the Hecke algebra $\mathbb{T}^D(\Gamma)$, of residue characteristic $l > 3$. Then there exists a prime q , not dividing $2NDl$, such that:*

1. $\bar{\rho}_{\mathfrak{m}}(\operatorname{Frob}_q)$ is not a scalar, and
2. *There is a maximal ideal $\tilde{\mathfrak{m}}$ of $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)^{q-\text{new}}$ such that the representations $\bar{\rho}_{\mathfrak{m}}$ and $\bar{\rho}_{\tilde{\mathfrak{m}}}$ are isomorphic.*

Proof: Let χ be the character such that $\det \bar{\rho}_{\mathfrak{m}} = \chi\omega$, where ω is the cyclotomic character. Let σ be the representation $\bar{\rho}_{\mathfrak{m}} \times \chi$. By the Čebotarev density theorem, we may choose q such that $\sigma(\operatorname{Frob}_q)$ is conjugate to $\sigma(c)$, where c denotes complex conjugation. Arguments of Diamond [Di] show that $\bar{\rho}_{\mathfrak{m}}(\operatorname{Frob}_q)$ then has order two, trace zero and determinant -1 . In this setting (1) is immediate; (2) follows from [DT], Theorem A. \blacksquare

Suppose we fix a q and $\tilde{\mathfrak{m}}$ as above. We will show that $\tilde{\mathfrak{m}}$ is q -old; that is, it descends to an ideal of the algebra $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)^{q\text{-old}}$. We first investigate the structure of this ring. The two degeneracy maps

$$J^D(\Gamma) \rightarrow J^D(\Gamma_0(q) \cap \Gamma)$$

induce an isogeny

$$J^D(\Gamma)^2 \rightarrow J^D(\Gamma_0(q) \cap \Gamma)_{q\text{-old}}.$$

This isogeny is compatible with a certain action of the Hecke operators on $J^D(\Gamma)^2$, in which the diamond bracket operators and the operators T_n for n prime to q act diagonally, and the action of T_q is given by the matrix

$$\begin{pmatrix} \tau_q & q\langle q \rangle \\ -1 & 0 \end{pmatrix}.$$

(Here τ_q denotes the endomorphism of $J^D(\Gamma)^2$ induced by the q th Hecke operator acting diagonally.) A proof may be found in [Di].

It follows that $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)^{q\text{-old}}$ is isomorphic to $R[T_q]/(T_q^2 - \tau_q T_q + q\langle q \rangle)$, where R is the subalgebra of $\mathbb{T}^D(\Gamma)$ generated by the diamond bracket operators and the Hecke operators T_n for n prime to q . By the lemma on p. 491 of [Wi], R is all of $\mathbb{T}^D(\Gamma)$. Moreover, the Eichler–Shimura relation, together with the fact that $\bar{\rho}_{\mathfrak{m}}$ and $\bar{\rho}_{\tilde{\mathfrak{m}}}$ are isomorphic, implies that $\pi(\tilde{\mathfrak{m}}) \cap R \subset \mathfrak{m}$, where π is the natural projection of $\mathbb{T}^D(\Gamma_0(q) \cap \Gamma)$ to its q -old quotient. In particular, $\tilde{\mathfrak{m}}$ is q -old as claimed.

PROPOSITION 7.2: *Let q and $\tilde{\mathfrak{m}}$ be chosen as above. Then*

$$\dim J^D(\Gamma_0(q) \cap \Gamma)[\tilde{\mathfrak{m}}] = \dim J^D(\Gamma_0(q) \cap \Gamma)_{q\text{-new}}[\tilde{\mathfrak{m}}] = \dim J^D(\Gamma_0(q) \cap \Gamma)_{q\text{-old}}[\tilde{\mathfrak{m}}].$$

Proof: To make the notation more concise, let $\tilde{J} = J^D(\Gamma_0(q) \cap \Gamma)$, $\tilde{J}_{\text{new}} = \tilde{J}_{q\text{-new}}$ and $\tilde{J}_{\text{old}} = \tilde{J}_{q\text{-old}}$. Let $J = J^D(\Gamma)$. As described above, the two degeneracy maps $J \rightarrow \tilde{J}$ induce a Hecke-equivariant isogeny $(J^2) \rightarrow \tilde{J}_{\text{old}}$. By [DT], Theorem 2, the kernel of this isogeny is Eisenstein; in particular it is supported away from $\tilde{\mathfrak{m}}$.

Let \tilde{J}^{old} be the q -old quotient of \tilde{J} (and thus the dual of \tilde{J}_{old}), and consider the sequence of maps

$$J^2 \rightarrow \tilde{J}_{\text{old}} \xrightarrow{i} \tilde{J} \xrightarrow{\psi} (\tilde{J})^\dagger \xrightarrow{i^\dagger} \tilde{J}^{\text{old}} \rightarrow J^2,$$

where the map ψ is given by Lemma 5.5, and the map $J^{\text{old}} \rightarrow J^2$ is the dual of the isogeny $J^2 \rightarrow \tilde{J}_{\text{old}}$. Calculations of Ribet [Ri1] show that the composition

of these maps is given by the matrix $\begin{pmatrix} q+1 & \langle q \rangle T_q \\ T_q & q+1 \end{pmatrix}$, and is in particular equal to $T_q^2 - \langle q \rangle$ times an automorphism of J^2 . Let $\psi_{\text{old}} : \tilde{J}_{\text{old}} \rightarrow \tilde{J}^{\text{old}}$ be the map $i^\dagger \psi i$, above. The kernel of ψ_{old} is equal to the intersection of \tilde{J}_{old} with the kernel of the map $\tilde{J} \rightarrow \tilde{J}^{\text{old}}$; the latter is precisely \tilde{J}_{new} . Since the intersection of \tilde{J}_{old} with \tilde{J}_{new} is finite, and \tilde{J}^{old} and \tilde{J}_{old} have the same dimension, ψ_{old} is surjective. Since the map $J^2 \rightarrow \tilde{J}_{\text{old}}$ is an isogeny, and $T_q^2 - \langle q \rangle : J^2 \rightarrow J^2$ factors as

$$J^2 \rightarrow \tilde{J}_{\text{old}} \xrightarrow{\psi_{\text{old}}} \tilde{J}^{\text{old}} \rightarrow J^2,$$

it follows that the map $T_q^2 - \langle q \rangle$ is a surjective endomorphism of J^2 . We immediately see that the map $T_q^2 - \langle q \rangle : \tilde{J}_{\text{old}} \rightarrow \tilde{J}_{\text{old}}$ is surjective as well. Finally, observe that since the kernel of the map $J^2 \rightarrow \tilde{J}_{\text{old}}$ is supported away from $\tilde{\mathfrak{m}}$; it follows that the $\tilde{\mathfrak{m}}$ -primary part of $\ker \psi_{\text{old}}$ is equal to the $\tilde{\mathfrak{m}}$ -primary part of $J_{\text{old}}[T_q^2 - \langle q \rangle]$.

The operator T_q acts on \tilde{J}_{new} as $-w_q$, where w_q is the Atkin–Lehner operator. Since $w_q^2 = \langle q \rangle$, it follows that $T_q^2 - \langle q \rangle$ annihilates \tilde{J}_{new} . The endomorphism of \tilde{J} induced by this map thus factors through \tilde{J}^{old} ; its image therefore lies in the q -old subvariety \tilde{J}_{old} . Since $(T_q^2 - \langle q \rangle)\tilde{J}_{\text{old}}$ is equal to \tilde{J}_{old} , this image must be all of \tilde{J}_{old} .

Now consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} & \longrightarrow & \tilde{J}_{\text{old}} \oplus \tilde{J}_{\text{new}} & \longrightarrow & \tilde{J} \longrightarrow 0 \\ & & \downarrow T_q^2 - \langle q \rangle & & \downarrow T_q^2 - \langle q \rangle & & \downarrow T_q^2 - \langle q \rangle \\ 0 & \longrightarrow & \tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} & \longrightarrow & \tilde{J}_{\text{old}} \oplus \tilde{J}_{\text{new}} & \longrightarrow & \tilde{J} \longrightarrow 0. \end{array}$$

The Snake lemma then yields an exact sequence:

$$0 \rightarrow \tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} \rightarrow \tilde{J}_{\text{old}}[T_q^2 - \langle q \rangle] \oplus \tilde{J}_{\text{new}} \rightarrow \tilde{J}[T_q^2 - \langle q \rangle] \rightarrow \tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} \rightarrow \tilde{J}_{\text{new}}.$$

The final map in this sequence is simply the inclusion $\tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} \rightarrow \tilde{J}_{\text{new}}$, and hence injective, so this yields a short exact sequence:

$$0 \rightarrow \tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}} \rightarrow \tilde{J}_{\text{old}}[T_q^2 - \langle q \rangle] \oplus \tilde{J}_{\text{new}} \rightarrow \tilde{J}[T_q^2 - \langle q \rangle] \rightarrow 0.$$

Taking $\tilde{\mathfrak{m}}^\infty$ -torsion then yields

$$0 \rightarrow (\tilde{J}_{\text{old}} \cap \tilde{J}_{\text{new}})_{\tilde{\mathfrak{m}}} \rightarrow \tilde{J}_{\text{old}}[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}} \oplus \tilde{J}_{\text{new}}[\tilde{\mathfrak{m}}^\infty] \rightarrow \tilde{J}[T_q^2 - \langle q \rangle, \tilde{\mathfrak{m}}^\infty] \rightarrow 0.$$

The discussion above showed that $J_{\text{old}}[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}}$ lies in the kernel of ψ_{old} , and hence lies in $J_{\text{new}}[T_q^2 - \langle q \rangle]$. It follows that $(J_{\text{old}} \cap J_{\text{new}})_{\tilde{\mathfrak{m}}} = J_{\text{old}}[T_q^2 - \langle q \rangle]_{\tilde{\mathfrak{m}}}$,

and hence that $\tilde{J}_{\text{new}}[\tilde{\mathfrak{m}}^\infty]$ is isomorphic to $\tilde{J}[T_q^2 - \langle q \rangle, \tilde{\mathfrak{m}}^\infty]$. Since $T_q^2 - \langle q \rangle$ lies in $\tilde{\mathfrak{m}}$ (since $\tilde{\mathfrak{m}}$ is q -new), it follows that $\tilde{J}_{\text{new}}[\tilde{\mathfrak{m}}] = \tilde{J}[\tilde{\mathfrak{m}}]$.

It remains to show that $\tilde{J}_{\text{old}}[\tilde{\mathfrak{m}}] = \tilde{J}[\tilde{\mathfrak{m}}]$. Suppose $\tilde{J}[\tilde{\mathfrak{m}}]$ has dimension $2n$. Consider the connected component \tilde{J}_q^0 of the special fiber of the Néron model of \tilde{J} at q . Since \tilde{J} has semistable reduction at q , it fits into an exact sequence

$$0 \rightarrow \text{Hom}(X_q(\tilde{J}), \mathbb{G}_m) \rightarrow \tilde{J}_q^0 \rightarrow \tilde{J}^2 \rightarrow 0.$$

Since the natural isogeny $J^2 \rightarrow \tilde{J}_{\text{old}}$ has kernel supported away from \mathfrak{m} , it suffices to show that $J^2[\tilde{\mathfrak{m}}]$ has dimension $2n$.

The new subvariety \tilde{J}_{new} has purely toric reduction at q , so the special fiber of its Néron model at q is an extension of the torus $\text{Hom}(X_q(\tilde{J}_{\text{new}}), \mathbb{G}_m)$ by the finite group $\Phi_q(\tilde{J}_{\text{new}})$. As $\bar{\rho}_{\mathfrak{m}}$ is unramified at q , and $\tilde{J}_{\text{new}}[\tilde{\mathfrak{m}}]$ has dimension $2n$, $(\tilde{J}_{\text{new}})_q[\tilde{\mathfrak{m}}]$ has dimension $2n$. We have an exact sequence:

$$0 \rightarrow \text{Hom}(X_q(\tilde{J}_{\text{new}}), \mathbb{G}_m)[\tilde{\mathfrak{m}}] \rightarrow (\tilde{J}_{\text{new}})_q[\tilde{\mathfrak{m}}] \rightarrow \Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}].$$

Moreover, since $\tilde{\mathfrak{m}}$ is controllable at q ($\bar{\rho}_{\mathfrak{m}}(\text{Frob}_q)$ is not a scalar), it follows, by Lemma 6.5 and Corollary 4.15, that $T_{\tilde{\mathfrak{m}}} \tilde{J}_{\text{new}} \cong X_q(\tilde{J}_{\text{new}})_{\tilde{\mathfrak{m}}}^2$. In particular, $X_q(\tilde{J}_{\text{new}})/\tilde{\mathfrak{m}}X_q(\tilde{J}_{\text{new}})$ has dimension n . The exact sequence above then shows that $\Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}]$ has dimension at least n . (We will see later that it must have dimension exactly n .)

The inclusion $\tilde{J}_{\text{new}} \rightarrow \tilde{J}$ induces an injection on the special fibers $(J_{\text{new}})_q$ and \tilde{J}_q of the corresponding Néron models, and in particular an isomorphism $X_q(\tilde{J}_{\text{new}}) \rightarrow X_q(J)$. Since the group of connected components $\Phi_q(\tilde{J})$ is Eisenstein (c.f. Proposition 5.6), $\Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}]$ maps into the connected component of \tilde{J}_q . Since the map on special fibers is injective, the image of $\Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}]$ in \tilde{J}_q has trivial intersection with the torus, and hence maps injectively into J^2 .

We thus obtain an injection of $\Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}]$ into $J^2[\tilde{\mathfrak{m}}]$. The latter is a direct sum of two dimensional subspaces V_i on which Frobenius acts via $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_q)$. Since Frob_q acts as a scalar on $\Phi_q(\tilde{J}_{\text{new}})$, the image of $\Phi_q(\tilde{J}_{\text{new}})$ in each V_i has dimension at most one. As $\Phi_q(\tilde{J}_{\text{new}})[\tilde{\mathfrak{m}}]$ has dimension at least n , it follows that $J^2[\tilde{\mathfrak{m}}]$ has dimension at least $2n$. But $\dim J^2[\tilde{\mathfrak{m}}] = \dim \tilde{J}_{\text{old}}[\tilde{\mathfrak{m}}]$, and the latter is less than $\dim \tilde{J}[\tilde{\mathfrak{m}}]$, which is $2n$. Thus $\dim \tilde{J}_{\text{old}}[\tilde{\mathfrak{m}}] = \dim \tilde{J}[\tilde{\mathfrak{m}}]$, as required. ■

We now return to the setting of the previous section; we have fixed a D dividing N that is the product of an even number of primes, a maximal ideal \mathfrak{m} , and a prime p dividing D , and we wish to show that

$$[J^{D,D}]_{\mathfrak{m}} \cong \text{Hom}(X_p(J^{\min}), X_p(J^{D,D}))_{\mathfrak{m}}.$$

Let p' be a prime dividing D other than p , and let $D' = D/(pp')$.

Now we choose two primes q_1 and q_2 as in Lemma 7.1. The choice of these primes determines several abelian varieties. In particular, the abelian varieties $J^{D'}(\Gamma_0(q_1 q_2 N/D') \cap \Gamma)$ and $J^D(\Gamma_0(q_1 q_2 N/D) \cap \Gamma)$ arise from $J^{D',D'}$ and $J^{D,D}$ by raising the level at q_1 and q_2 . We denote them by $\tilde{J}^{D',D'}$ and $\tilde{J}^{D,D}$, respectively. We will also need to consider the varieties

$$\tilde{J}^{D'q_1q_2,D'q_1q_2} = J^{D'q_1q_2}(\Gamma_0(N/D') \cap \Gamma)$$

and

$$\tilde{J}^{Dq_1q_2,Dq_1q_2} = J^{Dq_1q_2}(\Gamma_0(N/D) \cap \Gamma).$$

Let $\tilde{\mathcal{A}}^{D'}$, $\tilde{\mathcal{A}}^D$, $\tilde{\mathcal{A}}^{D'q_1q_2}$ and $\tilde{\mathcal{A}}^{Dq_1q_2}$ denote the categories of abelian varieties over \mathbb{Q} that are Hecke-equivariantly isogenous to $\tilde{J}^{D',D'}$, $\tilde{J}^{D,D}$, $\tilde{J}^{D'q_1q_2,D'q_1q_2}$ and $\tilde{J}^{Dq_1q_2,Dq_1q_2}$, respectively. Also consider the corresponding categories $(\tilde{\mathcal{A}}^{D'})^{\text{old}}$ and $(\tilde{\mathcal{A}}^D)^{\text{old}}$ associated to the $q_1 q_2$ -old quotients of $\tilde{J}^{D'}$ and \tilde{J}^D , respectively. We adopt all of the notational conventions introduced in Section 6 for these varieties as well; for instance, \tilde{J}^{D,Dq_1q_2} will denote the $q_1 q_2$ -new subvariety of $\tilde{J}^{D,D}$. Finally, let $\tilde{\mathfrak{m}}$ be a maximal ideal of $\tilde{\mathbb{T}}^{D'}$ whose associated Galois representation is isomorphic to $\bar{\rho}_{\mathfrak{m}}$.

The four degeneracy maps $\phi_1, \phi_{q_1}, \phi_{q_2}$ and $\phi_{q_1 q_2}$ from $J^{D',D'}$ to $\tilde{J}^{D',D'}$ induce an isogeny $(J^{D',D'})^4 \rightarrow (\tilde{J}^{D',D'})^{\text{old}}$, where the subscript old denotes the $q_1 q_2$ -old subvariety. The actions of T_{q_1} and T_{q_2} on $(J^{D',D'})^4$ are given by the matrices:

$$T_{q_1} = \begin{pmatrix} \tau_{q_1} & q_1 \langle q_1 \rangle & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \tau_{q_1} & q_1 \langle q_1 \rangle \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad T_{q_2} = \begin{pmatrix} \tau_{q_2} & 0 & q_2 \langle q_2 \rangle & 0 \\ 0 & \tau_{q_2} & 0 & q_2 \langle q_2 \rangle \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

where τ_{q_1} and τ_{q_2} denote the endomorphisms of $J^{D',D'}$ induced by the q_1 th and q_2 th Hecke operators. The diamond bracket operators, and the Hecke operators T_n for n prime to $q_1 q_2$, act diagonally. By the same arguments as those that follow the proof of Lemma 7.1, we have

$$(\tilde{\mathbb{T}}^{D'})^{\text{old}} \cong \mathbb{T}^{D'}[T_{q_1}, T_{q_2}] / (T_{q_1}^2 - \tau_{q_1} T_{q_1} + q_1 \langle q_1 \rangle, T_{q_2}^2 - \tau_{q_2} T_{q_2} + q_2 \langle q_2 \rangle).$$

In particular $(\tilde{\mathbb{T}}^{D'})^{\text{old}}$ is a free $\mathbb{T}^{D'}$ -algebra of rank 4.

More generally, given any abelian variety J in $\mathcal{A}^{D'}$ or \mathcal{A}^D , we can form the abelian variety J^4 on which $(\mathbb{T}^{D'})^{\text{old}}$ acts via the matrices given above. In this situation, we have

LEMMA 7.3: *Let A_1 and A_2 be abelian varieties in $\mathcal{A}^{D'}$ or \mathcal{A}^D . Then*

$$\mathrm{Hom}(A_1^4, A_2^4) \cong \mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{T}^{D'}} (\tilde{\mathbb{T}}^{D'})^{\mathrm{old}},$$

where Hom denotes morphisms in $(\tilde{\mathcal{A}}^{D'})^{\mathrm{old}}$ or $(\tilde{\mathcal{A}}^D)^{\mathrm{old}}$, respectively.

Proof: A map $A_1^4 \rightarrow A_2^4$ is given by a four by four matrix of maps $A_1 \rightarrow A_2$. For such a map to be $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ -equivariant, it must commute with the matrices defining the action of T_{q_1} and T_{q_2} . It is then straightforward to check that every such matrix can be written in the form $a^4 + T_{q_1}b^4 + T_{q_2}c^4 + T_{q_1}T_{q_2}d^4$ for maps a, b, c and d from A_1 to A_2 . (Here $a^4: (A_1)^4 \rightarrow (A_2)^4$ is the map obtained by evaluating the map a at each entry of the four-tuple defining a point of $(A_1)^4$.) That all maps of the above form commute with T_{q_1} and T_{q_2} , is clear. ■

Recall that we have fixed abelian varieties $J_{D'}^{\min}$ and J_D^{\min} in $\mathcal{A}^{D'}$ and \mathcal{A}^D that satisfy “multiplicity one” at all maximal ideals outside $S^{D'}$ or S^D . We will use the above results to make “compatible” choices of varieties J^{\min} for each of the other categories involved. The key is the following

LEMMA 7.4: *Let \mathfrak{m} be a maximal ideal of $\mathbb{T}^{D'}$ or \mathbb{T}^D , and $\tilde{\mathfrak{m}}$ a maximal ideal of $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ (or $(\tilde{\mathbb{T}}^D)^{\mathrm{old}}$) above \mathfrak{m} . Then $\dim_{\mathbb{T}^{D'}/\mathfrak{m}} J[\mathfrak{m}] = \dim_{(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}/\tilde{\mathfrak{m}}} (J^4)[\tilde{\mathfrak{m}}]$ (and analogously for \mathbb{T}^D and $(\tilde{\mathbb{T}}^D)^{\mathrm{old}}$).*

Proof: The set $\{1, T_{q_1}, T_{q_2}, T_{q_1}T_{q_2}\}$ is a basis for $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ over $\mathbb{T}^{D'}$. This basis gives rise to a dual basis for $\mathrm{Hom}_{\mathbb{T}^{D'}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}, \mathbb{T}^{D'})$, and the matrices giving the action of T_{q_1} and T_{q_2} with respect to this dual basis are the same as the matrices giving the action of T_{q_1} and T_{q_2} on $(J^{D'})^4$. It follows that $(J^{D'})^4[\mathfrak{m}]$ is isomorphic to $\mathrm{Hom}_{\mathbb{T}^{D'}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}, J^{D'}[\mathfrak{m}])$. Therefore, $(J^{D'})^4[\tilde{\mathfrak{m}}]$ is isomorphic to $\mathrm{Hom}_{\mathbb{T}^{D'}/\mathfrak{m}}((\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}/\tilde{\mathfrak{m}}, J^{D'}[\mathfrak{m}])$. The result follows immediately, and the proof for J^D is exactly the same. ■

It follows from this that $(J_{D'}^{\min})^4$ satisfies “multiplicity one” at every non-Eisenstein maximal ideal of $(\tilde{\mathbb{T}}^{D'})^{\mathrm{old}}$ of residue characteristic greater than 3. Denote this variety by $\tilde{J}_{D', \mathrm{old}}^{\min}$. By Theorem 4.11, the functor

$$A \mapsto \mathrm{Hom}_{\tilde{S}^{D'}}(A, \tilde{J}_{D', \mathrm{old}}^{\min})$$

gives an antiequivalence of categories from $(\tilde{\mathcal{A}}^{D'})_{\tilde{S}^{D'}}^{\mathrm{old}}$ to the category of “rank one” $(\tilde{\mathbb{T}}^{D'})_{\tilde{S}^{D'}}^{\mathrm{old}}$ -modules.

Let A_0 be any abelian variety in $\tilde{\mathcal{A}}^{D'}$ that satisfies “multiplicity one” at every non-Eisenstein maximal ideal of $\tilde{\mathbb{T}}^{D'}$ of residue characteristic greater than three.

Then the functor $A \mapsto \text{Hom}_{\tilde{S}^{D'}}(A_0, A)$ gives an antiequivalence of categories from $\tilde{\mathcal{A}}_{\tilde{S}^{D'}}^{D'}$ to the category of “rank one” $\tilde{\mathbb{T}}_{\tilde{S}^{D'}}^{D'}$ -modules. Moreover, the $q_1 q_2$ -old subvariety $(A_0)_{\text{old}}$ lies in $(\tilde{\mathcal{A}}^{D'})^{\text{old}}$ and satisfies “multiplicity one” outside of $\tilde{S}^{D'}$. In particular $\text{Hom}_{\tilde{S}^{D'}}(\tilde{J}_{D', \text{old}}^{\min}, (A_0)_{\text{old}})$ is an invertible $(\tilde{\mathbb{T}}^{D'})_{\tilde{S}^{D'}}^{\text{old}}$ -module M^{old} .

Let M be any invertible $\tilde{\mathbb{T}}_{\tilde{S}^{D'}}^{D'}$ -module such that $M \otimes_{\tilde{\mathbb{T}}_{\tilde{S}^{D'}}^{D'}} (\tilde{\mathbb{T}}^{D'})_{\tilde{S}^{D'}}^{\text{old}} \cong M^{\text{old}}$ and $\tilde{J}_{D'}^{\min}$ be an object of $\tilde{\mathcal{A}}^{D'}$ such that $\text{Hom}(A_0, \tilde{J}_{D'}^{\min}) \cong_{\tilde{S}^{D'}} M^{-1}$. Then $\text{Hom}(\tilde{J}_{D', \text{old}}^{\min}, (\tilde{J}_{D', \text{old}}^{\min})_{\text{old}})$ is a free $(\tilde{\mathbb{T}}^{D'})_{\tilde{S}^{D'}}^{\text{old}}$ -module of rank one. It follows that $\tilde{J}_{D', \text{old}}^{\min}$ and $\tilde{J}_{D'}^{\min}$ are “compatible” in the sense that $\tilde{J}_{D', \text{old}}^{\min}$ and $(\tilde{J}_{D'}^{\min})_{\text{old}}$ are $\tilde{S}^{D'}$ -isomorphic.

As usual, we take $\tilde{J}_D^{\min} = (\tilde{J}_{D'}^{\min})_{pp' - \text{new}}$, and similarly for $\tilde{J}_{D' q_1 q_2}^{\min}$ and $\tilde{J}_{D q_1 q_2}^{\min}$. Finally, to $(\tilde{\mathcal{A}}^D)^{\text{old}}$ we associate $\tilde{J}_{D, \text{old}}^{\min} = (\tilde{J}_D^{\min})_{\text{old}}$; this will be \tilde{S}^D -isomorphic to $(J_D^{\min})^4$. For each category $\tilde{\mathcal{A}}$ in the set

$$\{\tilde{\mathcal{A}}^{D'}, \tilde{\mathcal{A}}^D, \tilde{\mathcal{A}}^{D' q_1 q_2}, \tilde{\mathcal{A}}^{D q_1 q_2}, (\tilde{\mathcal{A}}^{D'})^{\text{old}}, (\tilde{\mathcal{A}}^D)^{\text{old}}\}$$

the above choice of abelian variety gives a functor $[-]_{\tilde{\mathcal{A}}}$ from $\tilde{\mathcal{A}}$ to the category of “rank one” modules over the corresponding Hecke algebra.

LEMMA 7.5: *In this setting we have:*

1. Let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^{D'}$ or $\tilde{\mathcal{A}}^D$. Then for any $J \in \tilde{\mathcal{A}}$, $[J]_{\tilde{\mathcal{A}}^{\text{old}}} \cong [J]_{\tilde{\mathcal{A}}} \otimes_{\tilde{\mathbb{T}}^{D'}} (\tilde{\mathbb{T}}^{D'})^{\text{old}}$.
2. Let $\mathcal{A} = \mathcal{A}^{D'}$ (resp. \mathcal{A}^D). Let $\tilde{\mathcal{A}}$ be $\tilde{\mathcal{A}}^{D'}$ (resp. $\tilde{\mathcal{A}}^D$). Then for any $J \in \mathcal{A}$, $[J^4]_{\tilde{\mathcal{A}}^{\text{old}}} = [J]_{\mathcal{A}} \otimes_{\mathbb{T}^{D'}} (\mathbb{T}^{D'})^{\text{old}}$.

Proof: We have a natural map $[J]_{\tilde{\mathcal{A}}} \rightarrow [J]_{\tilde{\mathcal{A}}^{\text{old}}}$ defined by restricting an element of $[J]$ to its $q_1 q_2$ -old subvariety. Locally at any $\tilde{\mathfrak{m}}$ outside $S_{\tilde{\mathcal{A}}}$, this map fits into a commutative diagram

$$\begin{array}{ccc} T_{\tilde{\mathfrak{m}}} J_{\tilde{\mathcal{A}}}^{\min} \otimes [J]_{\tilde{\mathcal{A}}} & \longrightarrow & T_{\tilde{\mathfrak{m}}} J \\ \downarrow & & \downarrow \\ T_{\tilde{\mathfrak{m}}} J_{\tilde{\mathcal{A}}^{\text{old}}}^{\min} \otimes [J^{\text{old}}]_{\tilde{\mathcal{A}}^{\text{old}}} & \longrightarrow & T_{\tilde{\mathfrak{m}}} J^{\text{old}}. \end{array}$$

Since J^{old} is a subvariety of J , the right-hand vertical map is surjective. But $T_{\tilde{\mathfrak{m}}} J_{\tilde{\mathcal{A}}^{\text{old}}}^{\min}$ is free of rank two, so the natural map $([J]_{\tilde{\mathcal{A}}})_{\tilde{\mathfrak{m}}} \rightarrow ([J^{\text{old}}]_{\tilde{\mathcal{A}}^{\text{old}}})_{\tilde{\mathfrak{m}}}$ is surjective as well. Statement (1) thus follows by Lemma 5.1. Statement (2) is immediate from the previous lemma. ■

With these technicalities out of the way, we begin our study of the geometry of these varieties. First of all, we have

LEMMA 7.6: *There are noncanonical isomorphisms:*

$$[\tilde{J}^{D,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \quad \text{and} \quad [\tilde{J}^{D',D'q_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{D'q_1q_2,D'q_1q_2}]_{\tilde{\mathfrak{m}}}.$$

Proof: We construct these isomorphisms for \tilde{J}^{D,Dq_1q_2} and $\tilde{J}^{Dq_1q_2,Dq_1q_2}$; the other case is nearly identical. Since $\tilde{J}^{Dq_1q_2,Dq_1q_2}$ is self-dual, we have

$$[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}}^*.$$

Since $\tilde{\mathfrak{m}}$ is controllable at q_1 , we have an isomorphism

$$[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong X_{q_1}(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}$$

by Proposition 4.14 and Lemma 6.5. On the other hand, by Proposition 5.7, we have an isomorphism

$$X_{q_1}(\tilde{J}^{Dq_1q_2,Dq_1q_2})_{\tilde{\mathfrak{m}}}^* \cong X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}.$$

Combining this with the previous two isomorphisms we find an isomorphism

$$[\tilde{J}^{Dq_1q_2,Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}.$$

As $\tilde{\mathfrak{m}}$ is controllable at q_2 , $X_{q_2}(\tilde{J}^{D,Dq_1q_2})_{\tilde{\mathfrak{m}}}$ is isomorphic to $[\tilde{J}^{D,Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ by Proposition 4.14 and Lemma 6.5. ■

PROPOSITION 7.7: *We have $\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$.*

Proof: By Lemma 7.6,

$$\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,Dq_1q_2}[\tilde{\mathfrak{m}}].$$

As \tilde{J}^{D,Dq_1q_2} is an abelian subvariety of $\tilde{J}^{D,D}$,

$$\dim \tilde{J}^{D,Dq_1q_2}[\tilde{\mathfrak{m}}] \leq \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}].$$

We therefore have

$$\dim \tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}] \leq \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}],$$

so it suffices to show that the opposite inequality holds.

Let Y_{q_1} denote the character group of $\tilde{J}^{Dq_1q_2,Dq_1q_2}[\tilde{\mathfrak{m}}]$ at q_1 , and X_{q_2} denote the character group of $J^D(\Gamma_0(q_2N/D) \cap \Gamma)^2$ at q_2 . Since $\tilde{\mathfrak{m}}$ is not Eisenstein and

has residue characteristic greater than 3, Proposition 5.9 implies that $(Y_{q_1}^*/Y_{q_1})_{\tilde{\mathfrak{m}}}$ is isomorphic to $(X_{q_2}/(T_{q_1}^2 - \langle q_1 \rangle)X_{q_2})_{\tilde{\mathfrak{m}}}$. In particular, we have

$$\dim Y_{q_1}^*/\tilde{\mathfrak{m}}Y_{q_1}^* \geq \dim Y_{q_1}^*/(Y_{q_1} + \tilde{\mathfrak{m}}Y_{q_1}^*) = \dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}.$$

We will relate $Y_{q_1}^*/\tilde{\mathfrak{m}}Y_{q_1}^*$ to $\tilde{J}^{Dq_1q_2, Dq_1q_2}[\tilde{\mathfrak{m}}]$, and $X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$ to $\tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$. The key is controllability of $\tilde{\mathfrak{m}}$ at q_1 and q_2 .

Since $\tilde{\mathfrak{m}}$ is controllable at q_1 , $[\tilde{J}^{Dq_1q_2, Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ is isomorphic to $(Y_{q_1})_{\tilde{\mathfrak{m}}}$ by Proposition 4.14 and Lemma 6.5. Thus $T_{\tilde{\mathfrak{m}}} \tilde{J}^{Dq_1q_2, Dq_1q_2}$ is isomorphic to $(Y_{q_1})_{\tilde{\mathfrak{m}}}^2$. It follows that $\tilde{J}^{Dq_1q_2, Dq_1q_2}[\tilde{\mathfrak{m}}]^\vee \cong (Y_{q_1}/\tilde{\mathfrak{m}}Y_{q_1})^2$. Since $\tilde{J}^{Dq_1q_2, Dq_1q_2}$ is self-dual, we also have $[\tilde{J}^{Dq_1q_2, Dq_1q_2}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{Dq_1q_2, Dq_1q_2}]_{\tilde{\mathfrak{m}}}^*$. In particular $(Y_{q_1})_{\tilde{\mathfrak{m}}}$ is also self-dual; that is, $(Y_{q_1})_{\tilde{\mathfrak{m}}} \cong (Y_{q_1}^*)_{\tilde{\mathfrak{m}}}$. We thus find that

$$\dim \tilde{J}^{Dq_1q_2, Dq_1q_2}[\tilde{\mathfrak{m}}] = 2 \dim Y_{q_1}/\tilde{\mathfrak{m}}Y_{q_1} = 2 \dim (Y_{q_1})^*/\tilde{\mathfrak{m}}(Y_{q_1})^*.$$

Combining this result with the above inequality, we see that it suffices to show that

$$\dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}] = 2 \dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}.$$

Since $\tilde{\mathfrak{m}}$ is controllable at q_2 , $(X_{q_2})_{\tilde{\mathfrak{m}}}^2$ is isomorphic to $T_{\tilde{\mathfrak{m}}} J^D(\Gamma_0(q_2N/D) \cap \Gamma)$, by Corollary 4.15. Thus

$$\dim J^D(\Gamma_0(q_2N/D) \cap \Gamma)[\tilde{\mathfrak{m}}] = 2 \dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}.$$

By [DT], Theorem 2, the isogeny

$$J^D(\Gamma_0(q_2N/D) \cap \Gamma)^2 \rightarrow (\tilde{J}^{D,D})_{q_2\text{-old}}$$

has kernel supported away from $\tilde{\mathfrak{m}}$, so $\dim(\tilde{J}^{D,D})_{q_2\text{-old}}[\tilde{\mathfrak{m}}] = 2 \dim X_{q_2}/\tilde{\mathfrak{m}}X_{q_2}$. But by Proposition 7.2,

$$\dim(\tilde{J}^{D,D})_{q_2\text{-old}}[\tilde{\mathfrak{m}}] = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}],$$

so the result follows. \blacksquare

If we fix an isomorphism: $[\tilde{J}^{D', D'q_1q_2}]_{\tilde{\mathfrak{m}}} \rightarrow [\tilde{J}^{D'q_1q_2, D'q_1q_2}]_{\tilde{\mathfrak{m}}}$, we obtain, by Theorem 4.11, an element of $\text{Hom}(\tilde{J}^{D'q_1q_2, D'q_1q_2}, \tilde{J}^{D', D'q_1q_2})_{\tilde{\mathfrak{m}}}$. Composing with the natural inclusion of $\tilde{J}^{D', D'q_1q_2}$ into $\tilde{J}^{D', D'}$, we obtain a map f in $\text{Hom}(\tilde{J}^{D'q_1q_2, D'q_1q_2}, \tilde{J}^{D', D'})_{\tilde{\mathfrak{m}}}$ which induces a surjection

$$[\tilde{J}^{D', D'}]_{\tilde{\mathfrak{m}}} \rightarrow [\tilde{J}^{D'q_1q_2, D'q_1q_2}]_{\tilde{\mathfrak{m}}}.$$

LEMMA 7.8: *There is a map g in $\text{Hom}(\tilde{J}^{Dq_1q_2, Dq_1q_2}, \tilde{J}^{D, D})_{\tilde{\mathfrak{m}}}$ whose dual g^\dagger fits in a commutative diagram:*

$$\begin{array}{ccc} X_p(\tilde{J}^{Dq_1q_2, Dq_1q_2})_{\tilde{\mathfrak{m}}} g^\dagger & \longrightarrow & X_{p'}(\tilde{J}^{D'q_1q_2, D'q_1q_2})_{\tilde{\mathfrak{m}}} \\ \downarrow & & \downarrow f^\dagger \\ X_p(\tilde{J}^{D, D})_{\tilde{\mathfrak{m}}} & \longrightarrow & X_{p'}(\tilde{J}^{D', D'})_{\tilde{\mathfrak{m}}}, \end{array}$$

in which the horizontal maps are those in the exact sequence of Theorem 5.4, the left-hand vertical map is induced by g^\dagger , and the right-hand vertical map is induced by f^\dagger .

Proof: Since the upper horizontal map in the above diagram identifies $X_p(\tilde{J}^{Dq_1q_2, Dq_1q_2})_{\tilde{\mathfrak{m}}}$ with $X_{p'}(\tilde{J}^{D, D})[I]$, where I is the kernel of the map $\tilde{\mathbb{T}} \rightarrow \tilde{\mathbb{T}}^{q_1q_2-\text{new}}$, and similarly for the lower horizontal map, it is clear that there is a map $X_p(\tilde{J}^{Dq_1q_2, Dq_1q_2})_{\tilde{\mathfrak{m}}} \rightarrow X_p(\tilde{J}^{D, D})_{\tilde{\mathfrak{m}}}$ making the above diagram commute; the only thing we have to check is that it is induced by an element g^\dagger of $\text{Hom}(\tilde{J}^{D, D}, \tilde{J}^{Dq_1q_2, Dq_1q_2})_{\tilde{\mathfrak{m}}}$. In particular it suffices to show that the natural map:

$$\begin{aligned} & \text{Hom}((\tilde{J}^{D, D})^{q_1q_2-\text{new}}, \tilde{J}^{Dq_1q_2, Dq_1q_2})_{\tilde{\mathfrak{m}}} \\ & \rightarrow \text{Hom}(X_p(\tilde{J}^{Dq_1q_2, Dq_1q_2}), X_p((\tilde{J}^{D, D})^{q_1q_2-\text{new}}))_{\tilde{\mathfrak{m}}} \end{aligned}$$

is an isomorphism. (Or, equivalently, that the natural map

$$\begin{aligned} & \text{Hom}([\tilde{J}^{Dq_1q_2, Dq_1q_2}], [(\tilde{J}^{D, D})^{q_1q_2-\text{new}}])_{\tilde{\mathfrak{m}}} \\ & \rightarrow \text{Hom}(X_p(\tilde{J}^{Dq_1q_2, Dq_1q_2}), X_p((\tilde{J}^{D, D})^{q_1q_2-\text{new}}))_{\tilde{\mathfrak{m}}} \end{aligned}$$

is an isomorphism.)

By Lemma 7.6, the modules $[\tilde{J}^{D, Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ and $[\tilde{J}^{Dq_1q_2, Dq_1q_2}]_{\tilde{\mathfrak{m}}}$ are isomorphic. It follows that there exists an injection of $[\tilde{J}^{D, Dq_1q_2}]$ into $[\tilde{J}^{Dq_1q_2, Dq_1q_2}]$ whose cokernel is supported away from $\tilde{\mathfrak{m}}$. By Theorem 4.11, this implies that there is an isogeny

$$\phi: \tilde{J}^{Dq_1q_2, Dq_1q_2} \rightarrow \tilde{J}^{D, Dq_1q_2}$$

whose kernel is supported away from $\tilde{\mathfrak{m}}$. Then ϕ^\dagger is an isogeny from $(\tilde{J}^{Dq_1q_2, Dq_1q_2})^\dagger$ to $(\tilde{J}^{D, Dq_1q_2})^\dagger$ whose kernel is supported away from $\tilde{\mathfrak{m}}$. But $\tilde{J}^{Dq_1q_2, Dq_1q_2}$ is self-dual, and

$$(\tilde{J}^{D, Dq_1q_2})^\dagger = (\tilde{J}_{q_1q_2-\text{new}}^{D, D})^\dagger \cong (\tilde{J}^{D, D})^{q_1q_2-\text{new}},$$

so we can consider ϕ^\dagger as an isogeny from $(\tilde{J}^{D,D})^{q_1 q_2 - \text{new}}$ to $\tilde{J}^{Dq_1 q_2, Dq_1 q_2}$, whose kernel is supported away from $\tilde{\mathfrak{m}}$. Viewed in this way, ϕ^\dagger induces isomorphisms between $[\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}$ and $[(\tilde{J}^{D,D})^{q_1 q_2 - \text{new}}]_{\tilde{\mathfrak{m}}}$, as well as between $X_p((\tilde{J}^{D,D})^{q_1 q_2 - \text{new}})_{\tilde{\mathfrak{m}}}$ and $X_p(\tilde{J}^{Dq_1 q_2, Dq_1 q_2})_{\tilde{\mathfrak{m}}}$.

Under these identifications, the natural map

$$\begin{aligned} \text{Hom}([\tilde{J}^{Dq_1 q_2, Dq_1 q_2}], [(\tilde{J}^{D,D})^{q_1 q_2 - \text{new}}])_{\tilde{\mathfrak{m}}} \\ \rightarrow \text{Hom}(X_p(\tilde{J}^{Dq_1 q_2, Dq_1 q_2}), X_p((\tilde{J}^{D,D})^{q_1 q_2 - \text{new}}))_{\tilde{\mathfrak{m}}} \end{aligned}$$

is identified with the map

$$\text{End}([\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}) \rightarrow \text{End}(X_p(\tilde{J}^{Dq_1 q_2, Dq_1 q_2})_{\tilde{\mathfrak{m}}}).$$

It thus suffices to show that the latter is an isomorphism. We have a sequence of isomorphisms:

$$\begin{aligned} \text{End}(X_p(\tilde{J}^{Dq_1 q_2, Dq_1 q_2})_{\tilde{\mathfrak{m}}}) \\ \cong \text{End}(X_p(J_{Dq_1 q_2}^{\min})_{\tilde{\mathfrak{m}}} \otimes [\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}) \\ \cong \text{Hom}([\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}, \text{Hom}(X_p(J_{Dq_1 q_2}^{\min}), X_p(\tilde{J}^{Dq_1 q_2, Dq_1 q_2}))_{\tilde{\mathfrak{m}}}) \\ \cong \text{Hom}([\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}, [\tilde{J}^{Dq_1 q_2, Dq_1 q_2}]_{\tilde{\mathfrak{m}}}). \end{aligned}$$

(The first isomorphism comes from Proposition 4.14; the second from the adjointness of Hom and tensor product, and the third from Corollary 6.6, using the fact that $\tilde{\mathfrak{m}}$ is controllable at q_1 .) It follows by Lemma 6.2 that the natural map in question is an isomorphism. ■

We are now in a position to prove

PROPOSITION 7.9: *The natural map $[\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \rightarrow \text{Hom}(X_p(\tilde{J}_D^{\min}), X_p(\tilde{J}^{D,D}))_{\tilde{\mathfrak{m}}}$ is an isomorphism.*

Proof: For conciseness of notation, we abbreviate $q_1 q_2$ by \underline{q} , and write X_p for $X_p(\tilde{J}_D^{\min})$ and $\overline{X_p}$ for $X_p(\tilde{J}_{D\underline{q}}^{\min})$. There is a natural surjection $\pi_p: X_p \rightarrow \overline{X_p}$ induced by the inclusion of $\tilde{J}_{D\underline{q}}^{\min}$ into \tilde{J}_D^{\min} .

We have a sequence of isomorphisms:

$$\text{Hom}(X_p, X_p(\tilde{J}^{D,D}))_{\tilde{\mathfrak{m}}}^* \rightarrow (X_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}}^* \rightarrow (X_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D',D})_{\tilde{\mathfrak{m}}},$$

where the first isomorphism comes from Lemma 2.5 and the second from Proposition 5.7. We also obtain isomorphisms

$$\text{Hom}(\overline{X_p}, X_p(\tilde{J}^{D\underline{q}, D\underline{q}}))_{\tilde{\mathfrak{m}}}^* \rightarrow (\overline{X_p})_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D\underline{q}, D\underline{q}})_{\tilde{\mathfrak{m}}}^* \rightarrow (\overline{X_p})_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D'\underline{q}, D\underline{q}})_{\tilde{\mathfrak{m}}},$$

in the same fashion. By Lemma 5.8 these fit into a commutative diagram:

$$\begin{array}{ccccc}
 \mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D}))_{\tilde{\mathfrak{m}}}^* & \longrightarrow & (X_p)_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}}^* & \longrightarrow & (X_p)_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D',D})_{\tilde{\mathfrak{m}}} \\
 (g^\dagger)^* \downarrow & & \pi_p \otimes (g^\dagger)^* \downarrow & & \pi_p \otimes \bar{f} \downarrow \\
 \mathrm{Hom}(\overline{X_p}, X_p(\tilde{J}^{Dq,Dq}))_{\tilde{\mathfrak{m}}}^* & \longrightarrow & (\overline{X_p})_{\tilde{\mathfrak{m}}} \otimes X_p(\tilde{J}^{Dq,Dq})_{\tilde{\mathfrak{m}}}^* & \longrightarrow & (\overline{X_p})_{\tilde{\mathfrak{m}}} \otimes X_{p'}(\tilde{J}^{D'q,Dq})_{\tilde{\mathfrak{m}}}^*.
 \end{array}$$

In particular, the right-hand vertical map is surjective, since the kernel of f is supported away from $\tilde{\mathfrak{m}}$, so the left-hand vertical map is surjective as well. This map fits into a commutative diagram:

$$\begin{array}{ccc}
 \mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D}))_{\tilde{\mathfrak{m}}}^* & \longrightarrow & [\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}}^* \\
 (g^\dagger)^* \downarrow & & (g^\dagger)^* \downarrow \\
 \mathrm{Hom}(\overline{X_p}, X_p(\tilde{J}^{Dq,Dq}))_{\tilde{\mathfrak{m}}}^* & \longrightarrow & [\tilde{J}^{Dq,Dq}]_{\tilde{\mathfrak{m}}}^*,
 \end{array}$$

where the horizontal maps are the \mathbb{Z}_l -duals of the canonical maps

$$[\tilde{J}^{D,D}] \rightarrow \mathrm{Hom}(X_p, X_p(\tilde{J}^{D,D})) \quad \text{and} \quad [\tilde{J}^{Dq,Dq}] \rightarrow \mathrm{Hom}(\overline{X_p}, X_p(\tilde{J}^{Dq,Dq})).$$

By Corollary 6.6, the bottom horizontal map is an isomorphism.

Since the left-hand vertical map is a surjection, the right-hand vertical map is as well. Moreover, since both $\tilde{J}^{D,D}$ and $\tilde{J}^{Dq,Dq}$ are self-dual, we have $[\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \cong [\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}}^*$ and similarly for $[\tilde{J}^{Dq,Dq}]$. By Proposition 7.7, and the fact that $2 \dim[\tilde{J}^{D,D}] \otimes \tilde{\mathbb{T}}/\tilde{\mathfrak{m}} = \dim \tilde{J}^{D,D}[\tilde{\mathfrak{m}}]$ (and similarly for $\tilde{J}^{Dq,Dq}$), it follows that the right-hand vertical map becomes an isomorphism after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$. Thus, after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$, the bottom horizontal map and the right-hand vertical maps are isomorphisms, and the left-hand vertical map is surjective. It follows that the upper horizontal map becomes surjective after tensoring with $\tilde{\mathbb{T}}/\tilde{\mathfrak{m}}$. By Nakayama's lemma, this implies that the upper horizontal map is surjective; since both of the modules in the top row have the same \mathbb{Z}_l -rank and no \mathbb{Z}_l -torsion, the upper horizontal map is an isomorphism. The result follows immediately. \blacksquare

It remains to pass from $\tilde{J}^{D,D}$ to $(\tilde{J}^{D,D})^{\mathrm{old}}$, and then to $J^{D,D}$.

PROPOSITION 7.10: *The natural map*

$$[(\tilde{J}^{D,D})^{\mathrm{old}}]_{\tilde{\mathfrak{m}}} \rightarrow \mathrm{Hom}(X_p(\tilde{J}_{D,\mathrm{old}}^{\min})_{\tilde{\mathfrak{m}}}, X_p((\tilde{J}^{D,D})^{\mathrm{old}})_{\tilde{\mathfrak{m}}})$$

is an isomorphism.

Proof: Let ϕ denote the natural map $\tilde{J}^{D,D} \rightarrow (\tilde{J}^{D,D})^{\text{old}}$. Composition with ϕ induces a natural map

$$[(\tilde{J}^{D,D})^{\text{old}}] = \text{Hom}((\tilde{J}^{D,D})^{\text{old}}, \tilde{J}_{D,\text{old}}^{\min}) \rightarrow \text{Hom}(\tilde{J}^{D,D}, \tilde{J}_{D,\text{old}}^{\min}).$$

Since every map $\tilde{J}^{D,D} \rightarrow \tilde{J}_{D,\text{old}}^{\min}$ factors through the $q_1 q_2$ -old quotient, this map is an isomorphism. The inclusion of $\tilde{J}_{D,\text{old}}^{\min}$ into \tilde{J}_D^{\min} thus induces a map $[(\tilde{J}^{D,D})^{\text{old}}] \rightarrow [\tilde{J}^{D,D}]$. Since $\tilde{J}_{D,\text{old}}^{\min}$ is the connected component of the identity in the subvariety of \tilde{J}_D^{\min} annihilated by the kernel I of the map $\tilde{\mathbb{T}}^D \rightarrow (\tilde{\mathbb{T}}^D)^{\text{old}}$, this map identifies $[(\tilde{J}^{D,D})^{\text{old}}]$ with the submodule of $[\tilde{J}^{D,D}]$ killed by I . It follows that the dual map $[\tilde{J}^{D,D}] \rightarrow [(\tilde{J}^{D,D})^{\text{old}}]$ is surjective.

We have a commutative diagram:

$$\begin{array}{ccc} \text{Hom}(X_p(\tilde{J}_D^{\min})_{\tilde{\mathfrak{m}}}, X_p(\tilde{J}^{D,D})_{\tilde{\mathfrak{m}}})^* & \xrightarrow{\quad} & [\tilde{J}^{D,D}]_{\tilde{\mathfrak{m}}} \\ \downarrow & & \downarrow \\ \text{Hom}(X_p(\tilde{J}_{D,\text{old}}^{\min})_{\tilde{\mathfrak{m}}}, X_p((\tilde{J}^{D,D})^{\text{old}})_{\tilde{\mathfrak{m}}})^* & \xrightarrow{\quad} & [(\tilde{J}^{D,D})^{\text{old}}]_{\tilde{\mathfrak{m}}}, \end{array}$$

in which the vertical maps are induced by the surjection $\tilde{J}^{D,D} \rightarrow (\tilde{J}^{D,D})^{\text{old}}$, and the horizontal maps are the \mathbb{Z}_l -duals of the maps taking a morphism of abelian varieties to the induced maps on character groups.

By the previous proposition, the upper horizontal map is an isomorphism; the right-hand vertical map is surjective. It follows that the lower horizontal map is surjective, and hence an isomorphism. ■

The following technical lemma is now all that we need in order to establish Theorem 6.1 at \mathfrak{m} .

LEMMA 7.11: *There is an isogeny $(\tilde{J}^{D,D})^{\text{old}} \rightarrow (J^{D,D})^4$ whose kernel is supported away from $\tilde{\mathfrak{m}}$.*

Proof: It is equivalent to show that $[(J^{D,D})^4]_{\tilde{\mathfrak{m}}} \cong [(\tilde{J}^{D,D})^{\text{old}}]_{\tilde{\mathfrak{m}}}$. The isogeny $(J^{D,D})^4 \rightarrow \tilde{J}_{\text{old}}^{D,D}$ has kernel supported away from $\tilde{\mathfrak{m}}$ by [DT], Theorem 2, so the dual isogeny $(\tilde{J}^{D,D})^{\text{old}} \rightarrow ((J^{D,D})^4)^\dagger$ has kernel supported away from $\tilde{\mathfrak{m}}$ as well. In particular,

$$[(\tilde{J}^{D,D})^{\text{old}}]_{\tilde{\mathfrak{m}}} \cong [((J^{D,D})^4)^\dagger]_{\tilde{\mathfrak{m}}} \cong [(J^{D,D})^4]_{\tilde{\mathfrak{m}}}^*,$$

so it suffices to show that $[(J^{D,D})^4]_{\tilde{\mathfrak{m}}}$ is self-dual.

We begin by showing that $(\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D)^* \cong (\mathbb{T}_{\tilde{\mathfrak{m}}}^D)^* \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D$. Observe that $\tilde{\mathbb{T}}^D$ is free of rank four over \mathbb{T}^D , with basis $\{1, T_{q_1}, T_{q_2}, T_{q_1} T_{q_2}\}$. This choice of basis allows

us to identify both $(\tilde{\mathbb{T}}^D)^*$ and $(\mathbb{T}^D)^* \otimes \tilde{\mathbb{T}}^D$ with $((\mathbb{T}^D)^*)^4$ as \mathbb{T}^D -modules. The Hecke operators T_{q_1} and T_{q_2} act via the matrices

$$T_{q_1} = \begin{pmatrix} 0 & -q_1 \langle q_1 \rangle & 0 & 0 \\ 1 & \tau_{q_1} & 0 & 0 \\ 0 & 0 & 0 & -q_1 \langle q_1 \rangle \\ 0 & 0 & 1 & \tau_{q_1} \end{pmatrix}; \quad T_{q_2} = \begin{pmatrix} 0 & 0 & -q_2 \langle q_2 \rangle & 0 \\ 0 & 0 & 0 & -q_2 \langle q_2 \rangle \\ 1 & 0 & \tau_{q_2} & 0 \\ 0 & 1 & 0 & \tau_{q_2} \end{pmatrix}$$

on the latter, and via the transposes of these matrices on the former. After tensoring with \mathbb{Z}_l , conjugation with the diagonal matrix

$$P = \begin{pmatrix} q_1 \langle q_1 \rangle q_2 \langle q_2 \rangle & 0 & 0 & 0 \\ 0 & -q_2 \langle q_2 \rangle & 0 & 0 \\ 0 & 0 & -q_1 \langle q_1 \rangle & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

sends the above matrices to their transposes. Thus P induces an isomorphism of $(\mathbb{T}^D)^* \otimes \tilde{\mathbb{T}}^D \otimes \mathbb{Z}_l$ with $(\tilde{\mathbb{T}}^D)^* \otimes \mathbb{Z}_l$. Localizing at $\tilde{\mathfrak{m}}$ then gives the desired isomorphism.

Now we have:

$$[(J^{D,D})^4]_{\tilde{\mathfrak{m}}}^* \cong \text{Hom}([J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D, (\tilde{\mathbb{T}}^D)_{\tilde{\mathfrak{m}}}^*) \cong \text{Hom}([J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D, (\mathbb{T}^D)_{\mathfrak{m}}^* \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D).$$

The latter is just $[J^{D,D}]_{\mathfrak{m}}^* \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D$, which is isomorphic to $[J^{D,D}]_{\mathfrak{m}} \otimes \tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D \cong [(J^{D,D})^4]_{\tilde{\mathfrak{m}}}$, as required. ■

Proof of Theorem 6.1: It suffices to prove that the natural map

$$[J^{D,D}]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min})_{\mathfrak{m}}, X_p(J^{D,D})_{\mathfrak{m}})$$

is an isomorphism. By Lemma 7.11 and Proposition 7.10, the natural map:

$$[J^{D,D}]_{\tilde{\mathfrak{m}}}^4 \rightarrow \text{Hom}(X_p((J_D^{\min})^4)_{\tilde{\mathfrak{m}}}, X_p((J^{D,D})^4)_{\tilde{\mathfrak{m}}})$$

is an isomorphism. It is easy to check that this latter map is just the corresponding map

$$[J^{D,D}]_{\mathfrak{m}} \rightarrow \text{Hom}(X_p(J_D^{\min})_{\mathfrak{m}}, X_p(J^{D,D})_{\mathfrak{m}})$$

tensoring over $\mathbb{T}_{\mathfrak{m}}^D$ with $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D$. Since $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}^D$ is free over $\mathbb{T}_{\mathfrak{m}}^D$, the result is immediate. ■

8. Main results

Having successfully proven Theorem 6.1, we will now make use of it to compute $[J^D]$ for D dividing N . Recall that for each D dividing N with an even number of prime divisors, we have constructed a locally-free $\mathbb{T}_{S^D}^D$ -module L_D such that $[(J_D^{\min})^\dagger] \cong_{S^D} (\mathbb{T}^D)^* \otimes L_D$, which is unique up to isomorphism (Proposition 4.17). By Proposition 4.17 and Lemma 5.5, we have an isomorphism: $[J^{D,D}] \cong_{S^D} [J^{D,D}]^* \otimes L_D$.

PROPOSITION 8.1: *Let p and q be two primes dividing D . Then we have*

$$[J^{D,D}] \cong_{S^D} [J^{\frac{D}{pq},D}] \otimes X_p(J_D^{\min}) \otimes X_q(J_D^{\min}) \otimes L_D.$$

Proof: We have a sequence of isomorphisms:

$$\begin{aligned} [J^{D,D}] &\cong_{S^D} [J^{D,D}]^* \otimes L_D \\ &\cong_{S^D} \operatorname{Hom}(X_p(J_D^{\min}), X_p(J^{D,D}))^* \otimes L_D && \text{(Theorem 6.1)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_p(J^{D,D})^* \otimes L_D && \text{(Lemma 2.5)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_q(J^{\frac{D}{pq},D}) \otimes L_D && \text{(Proposition 5.7)} \\ &\cong_{S^D} X_p(J_D^{\min}) \otimes X_q(J_D^{\min}) \otimes [J^{\frac{D}{pq},D}] \otimes L_D && \text{(Proposition 4.14),} \end{aligned}$$

which immediately proves the claim. \blacksquare

To extend this to a relationship between $[J^{D,R}]$ and $[J^{Dpq,R}]$ for R divisible by Dpq , we use the following lemma:

LEMMA 8.2: *Let D , R , and R' be divisors of N such that D is divisible by an even number of primes, and such that $D|R|R'$. Then $[J^{D,R'}] \cong_{S_{R'}} [J^{D,R}] \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

Proof: There is a natural map $[J^{D,R}] \rightarrow [J^{D,R'}]$ given by taking an element of $[J^{D,R}]$, treating it as an element of $\operatorname{Hom}(J^{D,R}, J_R^{\min})$, and restricting it to $J^{D,R'} \subset J^{D,R}$. (The image of $J^{D,R'}$ will always land in $J_{R'}^{\min} \subset J_R^{\min}$.) By Lemma 5.1, it suffices to show the surjectivity of this map at \mathfrak{m} outside $S_{R'}$. Let $\hat{\mathfrak{m}}$ be the preimage of \mathfrak{m} in \mathbb{T}^R . We have a commutative diagram:

$$\begin{array}{ccc} T_{\hat{\mathfrak{m}}} J_R^{\min} \otimes [J^{D,R}] & \longrightarrow & T_{\hat{\mathfrak{m}}} J_{R'}^{\min} \otimes [J^{D,R'}]_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ T_{\hat{\mathfrak{m}}} J^{D,R} & \longrightarrow & T_{\hat{\mathfrak{m}}} J^{D,R'}, \end{array}$$

in which the vertical maps are isomorphisms. The lower horizontal map is induced by an inclusion of varieties and is therefore surjective. The upper

horizontal map is the restriction map we wish to study, tensored with $T_{\mathfrak{m}} J_R^{\min}$. The surjectivity of the lower horizontal map means that the upper horizontal map is surjective as well. But since $T_{\mathfrak{m}} J_R^{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}^R$, this implies that the map $[J^{D,R}]_{\mathfrak{m}} \rightarrow [J^{D,R'}]_{\mathfrak{m}}$ is surjective as well. ■

We also need to relate L_R with $L_{R'}$ for R dividing R' ; since we have chosen our J_R^{\min} compatibly, we have the following

LEMMA 8.3: *If R divides R' , then $L_{R'}$ is $S^{R'}$ -isomorphic to $L_R \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$.*

Proof: The property

$$[(J_{R'}^{\min})^{\dagger}] \cong_{S^{R'}} (\mathbb{T}^{R'})^* \otimes L_{R'}$$

characterises R' up to $S^{R'}$ -isomorphism. Since $J_{R'}^{\min}$ is the $\frac{R'}{R}$ -new subvariety of J_R^{\min} , $J_{R'}^{\min}$ is the connected component of $J_R^{\min}[I]$, where I is the kernel of the map $\mathbb{T}^R \rightarrow \mathbb{T}^{R'}$. Thus $[(J_{R'}^{\min})^{\dagger}] = \text{Hom}((J_{R'}^{\min})^{\dagger}, J_{R'}^{\min})$ is isomorphic to $\text{Hom}((J_{R'}^{\min})^{\dagger}, J_R^{\min})[I]$. The injection $J_{R'}^{\min} \rightarrow J_R^{\min}$ dualizes to a surjection $(J_R^{\min})^{\dagger} \rightarrow (J_{R'}^{\min})^{\dagger}$; moreover, any morphism of $(J_R^{\min})^{\dagger}$ that is killed by I factors through $(J_{R'}^{\min})^{\dagger}$. Thus

$$\text{Hom}((J_{R'}^{\min})^{\dagger}, J_R^{\min})[I] \cong_{S^{R'}} \text{Hom}((J_R^{\min})^{\dagger}, J_R^{\min})[I] = [(J_R^{\min})^{\dagger}][I].$$

It follows that $[(J_{R'}^{\min})^{\dagger}] \cong [(J_R^{\min})^{\dagger}][I]$. Dualizing, and invoking Corollary 5.3, we find that $L_{R'}^{-1} \cong L_R \otimes_{\mathbb{T}^R} \mathbb{T}^{R'}$, as required. ■

Putting these together, and passing to \mathcal{A}^N , we find;

PROPOSITION 8.4: *Let D divide N and be divisible by an even number of primes. For any p and q dividing D , we have:*

$$[J^D] \cong_{S^D} [J^{D/(pq)}] \otimes X_p(J^{\min}) \otimes X_q(J^{\min}) \otimes L_N.$$

Proof: Tensor the isomorphism of Proposition 8.1 with \mathbb{T}^N and apply Lemmas 8.2 and 8.3. ■

Working inductively, one immediately proves

THEOREM 8.5: *For any D dividing N and divisible by $2k$ primes, we have:*

$$[J^D] \cong_S [J^1] \otimes (L_N)^{\otimes k} \otimes \bigotimes_{p|D} X_p(J^{\min}).$$

Combining this with Proposition 4.12, we obtain:

COROLLARY 8.6: *Let D and D' be divisors of N , divisible by $2n$ and $2m$ primes, respectively, such that D divides D' . Let M be the module*

$$(L_N)^{\otimes m-n} \otimes \bigotimes_{p \mid \frac{D'}{D}} X_p(J^{\min}).$$

Then there is an S -isomorphism:

$$g: \operatorname{Hom}(J^D, J^{D'}) \rightarrow \operatorname{Hom}(M, \operatorname{End}(J^D))$$

such that if $\phi \in \operatorname{Hom}_S(J^D, J^{D'})$, and $I_\phi \subset \operatorname{End}(J^D)$ is the ideal generated by image of M under $g(\phi): M \rightarrow \operatorname{End}(J^D)$, then $\ker_S \phi = J^D[I_\phi]$ up to support on S .

This result determines the S -isomorphism class of $J^{D'}$ in terms of J^D ; in particular, if one has J^D , and one wishes to construct an abelian variety S -isomorphic to $J^{D'}$, one merely has to find an ideal I of \mathbb{T} (or, more generally, of $\operatorname{End}(J_D)$) isomorphic to M , and construct $J^D/J^D[I]$. Unfortunately, although M is independent of the choice of J^{\min} , the various factors in the product defining M are not, making M difficult to compute. One can remedy this, however, by introducing a suitable “multiplicity one” hypothesis.

In particular, we expand our set S to include those maximal ideals for which $J^{1,1} = J^1(\Gamma_0(N) \cap \Gamma)$ fails to satisfy “multiplicity one”; that is, we take S_+^R to be the set of $\mathfrak{m} \subset \mathbb{T}^R$ such that either \mathfrak{m} is in S^R or $J^1(\Gamma_0(N) \cap \Gamma)[\mathfrak{m}]$ has dimension greater than two. (The latter is expected to be a very rare occurrence; in particular it cannot occur if l is prime to $2NM$ [Ti]. Moreover, the only known counterexamples are in residue characteristic 2 [Ki].) As usual we take $S_+ = S_+^N$.

The advantage of enlarging our set of “bad primes” to S_+ is that we can take $J_R^{\min} = J^{1,R}$ for all R ; the “multiplicity one” condition we have imposed above makes this a valid choice. Moreover, we can compute L_N in this setting. Since $J_\chi^{1,1} \cong (J^{1,1})^\vee$, and $[J^{1,1}]_\chi \cong_{S_+^1} \mathbb{T}^1$, we have $\mathbb{T}^1 \cong_{S_+^1} L^1 \otimes (\mathbb{T}^1)^*$. We thus recover the result that $\mathbb{T}_{S_+^1}^1$ is Gorenstein, and find that $L^1 \cong_{S_+^1} ((\mathbb{T}_{S_+^1}^1)^*)^{-1}$. Lemma 8.3 then determines L_N explicitly. Putting this together, we find:

THEOREM 8.7: *For any D dividing N and divisible by exactly $2k$ primes, we have*

$$[J^D] \cong_{S_+} (L_N)^{\otimes k} \otimes \bigotimes_{p \mid D} X_p(J^1).$$

COROLLARY 8.8: *Let D and D' be divisors of N , divisible by exactly $2n$ and $2m$ primes, respectively, such that D divides D' . Let M be the module*

$$(L_N)^{\otimes m-n} \otimes \bigotimes_{p \mid \frac{D'}{D}} X_p(J^1).$$

Then there is an S_+ -isomorphism:

$$g: \operatorname{Hom}(J^D, J^{D'}) \rightarrow \operatorname{Hom}(M, \operatorname{End}(J^D))$$

such that if $\phi \in \operatorname{Hom}_{S_+}(J^D, J^{D'})$, and $I_\phi \subset \operatorname{End}(J^D)$ is the ideal generated by image of M under $g(\phi): M \rightarrow \operatorname{End}(J^D)$, then $\ker_{S_+} \phi = J^D[I_\phi]$ up to support on S_+ .

Remark 8.9: The previous two results, although valid over a smaller subset of $\operatorname{Spec} \mathbb{T}$, have the advantage that the character groups that appear within them are explicitly computable, via the techniques in [Ko] together with Corollary 5.3. It is thus possible, given the isomorphism class of J^1 , to *effectively compute* an abelian variety which is S_+ -isomorphic to J^D for any D one desires.

The following result, a strengthening of a result proved by Yang [Ya] in the case in which $\bar{\rho}_{\mathfrak{m}}$ is ramified at at least half of the primes dividing D , is an immediate corollary.

COROLLARY 8.10: *Let \mathfrak{m} be a maximal ideal of \mathbb{T} outside S , and let k be the number of primes dividing D at which \mathfrak{m} is not controllable. Then $\dim J^D[\mathfrak{m}] \leq 2^k \dim J^1[\mathfrak{m}]$.*

Proof: Since $J^D[\mathfrak{m}]$ is naturally isomorphic to $(T_{\mathfrak{m}}J^D/\mathfrak{m}T_{\mathfrak{m}}J^D)^\vee$, and similarly for $J^1[\mathfrak{m}]$, it suffices to show that

$$\dim T_{\mathfrak{m}}J^D/\mathfrak{m}T_{\mathfrak{m}}J^D \leq 2^k \dim J^1/\mathfrak{m}T_{\mathfrak{m}}J^1.$$

Moreover, we have isomorphisms

$$T_{\mathfrak{m}}J^D \cong [J^D]_{\mathfrak{m}} \otimes T_{\mathfrak{m}}J^{\min},$$

and similarly for $T_{\mathfrak{m}}J^1$, so it suffices to show that

$$\dim[J^D]/\mathfrak{m}[J^D] \leq 2^k \dim[J^1]/\mathfrak{m}[J^1].$$

By Nakayama's lemma, this is equivalent to showing that the size of a minimal generating set for $[J^D]_{\mathfrak{m}}$ is at most 2^k times the size of a minimal generating set for $[J^1]_{\mathfrak{m}}$.

By Theorem 8.5, we have

$$[J^D]_{\mathfrak{m}} \cong [J^1]_{\mathfrak{m}} \otimes \bigotimes_{p|D} X_p(J^{\min})_{\mathfrak{m}}.$$

(The factors of L_N are locally free and thus disappear after localizing at \mathfrak{m} .) If \mathfrak{m} is controllable at p , $X_p(J^{\min})_{\mathfrak{m}}$ is free of rank one, and can be ignored. If \mathfrak{m} is not controllable at p , the surjection $T_{\mathfrak{m}}J^{\min} \rightarrow X_p(J^{\min})_{\mathfrak{m}}$ constructed in Proposition 4.14 shows that $X_p(J^{\min})_{\mathfrak{m}}/\mathfrak{m}X_p(J^{\min})_{\mathfrak{m}}$ has dimension at most two. In particular, $X_p(J^{\min})_{\mathfrak{m}}$ is generated by a two-element set, so tensoring with $X_p(J^{\min})_{\mathfrak{m}}$ increases the size of a minimal generating set by at most a factor of two. The result follows. ■

Considering the special case in which \mathfrak{m} is controllable at *every* prime dividing D yields a “multiplicity one” result for Jacobians of Shimura curves:

COROLLARY 8.11: *Let \mathfrak{m} be a maximal ideal of \mathbb{T} lying outside S . If \mathfrak{m} is controllable at every prime dividing D , then $J^D[\mathfrak{m}]$ and $J^1[\mathfrak{m}]$ have the same dimension. In particular, if \mathfrak{m} satisfies “multiplicity one” for $J^{1,1} = J^1(\Gamma_0(N) \cap \Gamma)$, then $\dim J^D[\mathfrak{m}] = 2$.*

Proof: By Theorem 8.5, and the fact that $X_p(J^{\min})_{\mathfrak{m}}$ is free for every p dividing D , we have $[J^D]_{\mathfrak{m}} \cong [J^1]_{\mathfrak{m}}$. It follows that $T_{\mathfrak{m}}J^D \cong T_{\mathfrak{m}}J^1$, and hence that $J^D[\mathfrak{m}] \cong J^1[\mathfrak{m}]$. If \mathfrak{m} satisfies “multiplicity one” for the full Jacobian $J^{1,1}$, then it also satisfies “multiplicity one” for J^1 , as the latter is just the N -new subvariety of the former. In particular $J^1[\mathfrak{m}]$ has dimension two and the result follows. ■

Remark 8.12: Although we have been working throughout with maximal ideals \mathfrak{m} outside S (and hence outside of residue characteristic 2 or 3), the preceding corollary holds in more generality. In particular, if $\Gamma = \Gamma_0(M)$ or if Γ contains $\Gamma_1(r)$, for $r \geq 4$, then the preceding corollary holds for all non-Eisenstein \mathfrak{m} . To establish this, one observes that the only places in the above argument in which we need the maximal ideals of residue characteristic 2 or 3 to lie in S are in Proposition 5.6 and in the level-raising arguments of Section 6. Since the latter are unnecessary for an \mathfrak{m} that is already controllable at every prime dividing D , and Proposition 5.6 holds even in residue characteristics 2 and 3 when $\Gamma = \Gamma_0(M)$ or $\Gamma_1(M)$ 5.7, all of the results we have obtained above hold locally at \mathfrak{m} . This is enough to establish the “multiplicity one” result above.

We also have the following alternative characterization of the relationship between $[J^D]$ and $[J^{D'}]$ for D dividing D' :

PROPOSITION 8.13: *Let D and D' be divisors of N , each divisible by an even number of primes, and suppose that D divides D' . Then*

$$\mathrm{Hom}(J^{D',D'}, J^{D,D'}) \cong_{S^{D'}} \mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'}))$$

for any p dividing D' .

Proof: We have an isomorphism

$$X_p(J^{D,D'}) \cong_{S^{D'}} X_p(J_{D'}^{\min}) \otimes [J^{D,D'}].$$

This induces an isomorphism

$$\mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})) \cong_{S^{D'}} \mathrm{Hom}([J^{D,D'}], \mathrm{Hom}(X_p(J_{D'}^{\min}), X_p(J^{D',D'}))),$$

by the adjointness of Hom and tensor products. By Theorem 6.1, the latter module is naturally $S^{D'}$ -isomorphic to $\mathrm{Hom}([J^{D,D'}], [J^{D',D'}])$, and hence to $\mathrm{Hom}(J^{D',D'}, J^{D,D'})$ by Theorem 4.11. ■

COROLLARY 8.14: *We have a natural isomorphism*

$$[J^{D',D'}] \cong_{S^{D'}} [J^{D,D'}] \otimes \mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})).$$

Proof: By Theorem 8.5, there is an M such that $[J^{D',D'}] \cong_{S^{D'}} M \otimes [J^{D,D'}]$. By Lemma 6.2, it follows that $[J^{D',D'}] \cong_{S^{D'}} \mathrm{Hom}([J^{D,D'}], [J^{D',D'}]) \otimes [J^{D,D'}]$. The result follows by Proposition 8.13. ■

COROLLARY 8.15: *Fix D and D' as above, and let p divide D' . Let M be the module*

$$\mathrm{Hom}(X_p(J^{D,D'}), X_p(J^{D',D'})) \otimes_{\mathbb{T}^{D'}} \mathbb{T}.$$

Then $[J^{D'}] = M \otimes [J^D]$, and there is an S -isomorphism:

$$g: \mathrm{Hom}(J^D, J^{D'}) \rightarrow \mathrm{Hom}(M, \mathrm{End}(J^D))$$

such that if $\phi \in \mathrm{Hom}_S(J^D, J^{D'})$, and $I_\phi \subset \mathrm{End}(J^D)$ is the ideal generated by image of M under $g(\phi): M \rightarrow \mathrm{End}(J^D)$, then $\ker_S \phi = J^D[I_\phi]$ up to support on S .

Proof: Immediate from Corollary 8.14, Lemma 8.2, and Proposition 4.12. ■

9. Further questions

These results leave several questions unanswered. The most obvious of these is whether or not the set S can be made smaller. At the moment, a maximal ideal \mathfrak{m} can be in S either because it is Eisenstein or because it has a residue characteristic 2 or 3.

It seems likely that in many cases the non-Eisenstein maximal ideals of residue characteristic 2 and 3 may be removed from S . At the moment there are two places in the argument that require the residue characteristic to be greater than 3. The first is that component groups of Jacobians of some Shimura curves can have support at non-Eisenstein primes of residue characteristic 2 or 3, which interferes with the proof of Proposition 5.7. This does not happen, however, if Γ either has the form $\Gamma_0(M)$ for some M , or if Γ contains $\Gamma_1(r)$ for some $r \geq 4$ [Ri4]. Thus in many cases this does not pose a problem.

The more serious difficulty occurs in Section 7, where we rely on results of [DT] that only hold in residue characteristic greater than 3. The only result which we really need from that paper is Theorem 2 (we invoke Theorem A at one point but if one has Theorem 2 then the case we need of Theorem A seems to follow quickly.) Moreover, since we have the luxury of choosing the primes at which we raise the level, we need only find two primes q_1 and q_2 for which Theorem 2 holds locally at \mathfrak{m} . It seems very unlikely that this will not happen. It is thus reasonable to conjecture that (when Γ satisfies one of the conditions given above) Theorem 6.1 holds even when S contains only Eisenstein primes.

The non-Eisenstein condition, on the other hand, is much more serious. All of the techniques of section 4, and Proposition 5.7, fail to hold at Eisenstein maximal ideals. Thus applying the techniques of this paper in this case seems hopeless.

We have also left open the question of whether or not there exist *canonical* maps between the varieties in question. If one is interested in looking for such things, one place to start would be to try to construct canonical maps between the modules constructed in Section 8.

Finally, Corollary 8.11 provides a sufficient condition for “multiplicity one” for Jacobians of Shimura curves. It is interesting to ask if this condition is necessary as well. This would hold if one could establish a converse to Lemma 6.5. A special case of such a result appears in [Ri2], where Ribet constructs character groups that are not free of rank one at specific maximal ideals of the Hecke algebra.

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10. Appendix: The Deligne-Rapoport Model for Shimura curves

Several of the proofs in Section 6 rely on the existence of a certain model for Shimura curves over a prime of bad reduction. Buzzard [Bu1] constructed such a model in the case where the Shimura curve in question is a fine moduli space; i.e., when the congruence subgroup Γ giving the level structure is sufficiently small. Here we extend Buzzard's results to include the case when the Shimura curve is only a coarse moduli space.

Let D be a squarefree product of an even number of primes, and let B be an indefinite quaternion algebra over \mathbb{Q} of discriminant D . Let \mathcal{O} be a maximal order of B . Fix a congruence subgroup Γ of level N prime to D , and consider the Shimura curves $X = X^D(\Gamma)$, and $X_p = X^D(\Gamma_0(p) \cap \Gamma)$, for p prime to ND , as schemes over $\mathbb{Z}[\frac{1}{ND}]$.

We first assume that Γ is contained in $\Gamma_1(r)$ for some $r > 3$. In this case (see for instance [Bu1]) the curves X and X_p are fine moduli spaces for “false elliptic curves” for B with level structure. More precisely:

Definition 10.1: A false elliptic curve for \mathcal{O} is an abelian surface A , together with an action of \mathcal{O} on A , such that for all $\tau \in \mathcal{O}$, the reduced trace of τ is equal to the trace of τ acting on the tangent space to A at the identity.

Then X is the fine moduli space parameterizing false elliptic curves for \mathcal{O} with Γ -level structure, and X_p is the fine moduli space parametrizing triples (A, P, ρ) , where A is a false elliptic curve for \mathcal{O} , P is a Γ -level structure on A , and $\rho: A \rightarrow B$ is an \mathcal{O} -equivariant isogeny of false elliptic curves, of degree p^2 .

When we restrict to the fiber over $\overline{\mathbb{F}}_p$, we obtain two morphisms

$$\text{Frob}, \text{Ver}: X_{\overline{\mathbb{F}}_p} \rightarrow (X_p)_{\overline{\mathbb{F}}_p}.$$

The former takes a pair (A, P) to the triple $(A, P, \text{Frob}: A \rightarrow A^{(p)})$, and the latter takes (A, P) to the triple $(A^{(p)}, P^{(p)}, \text{Ver}: A^{(p)} \rightarrow A)$. Then one has:

THEOREM 10.2 (Buzzard, [Bu1]):

1. X and X_p are regular schemes, defined over $\mathbb{Z}[\frac{1}{ND}]$.
2. X is smooth over $\mathbb{Z}[\frac{1}{ND}]$, and X_p is smooth over $\mathbb{Z}[\frac{1}{NDp}]$.

3. The maps Frob and Ver are closed immersions, and their images are the two irreducible components of $(X_p)_{\overline{\mathbb{F}}_p}$. These two components intersect transversely at the supersingular points of $(X_p)_{\overline{\mathbb{F}}_p}$.
4. At a supersingular point x of $(X_p)_{\overline{\mathbb{F}}_p}$, the completion of the local ring is isomorphic to

$$W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p).$$

In the case where Γ is arbitrary, X and X_p need not be fine moduli spaces. Let \mathcal{X} (resp. \mathcal{X}_p) denote the moduli stack for pairs (A, P) where A is a false elliptic curve for \mathcal{O} and P is a Γ -level structure on A (resp. triples (A, P, ρ) where A and P are as before and ρ is an \mathcal{O} -equivariant isogeny of degree p^2). Then we can define maps $\text{Frob}, \text{Ver}: \mathcal{X}_{\overline{\mathbb{F}}_p} \rightarrow (\mathcal{X}_p)_{\overline{\mathbb{F}}_p}$ exactly as before. These maps induce maps $\text{Frob}, \text{Ver}: X_{\overline{\mathbb{F}}_p} \rightarrow (X_p)_{\overline{\mathbb{F}}_p}$, as X and X_p are the underlying coarse moduli spaces of \mathcal{X} and \mathcal{X}_p . We then have:

THEOREM 10.3: *For arbitrary congruence subgroups Γ , one has:*

1. X is a regular scheme over $\mathbb{Z}[\frac{1}{ND}]$; X_p is defined over $\mathbb{Z}[\frac{1}{ND}]$ and regular away from the supersingular points on the fiber over $\overline{\mathbb{F}}_p$.
2. X is smooth over $\mathbb{Z}[\frac{1}{ND}]$, and X_p is smooth over $\mathbb{Z}[\frac{1}{NDp}]$.
3. The maps Frob and Ver are closed immersions, and their images are the two irreducible components of $(X_p)_{\overline{\mathbb{F}}_p}$. These two components intersect transversely at the supersingular points of $(X_p)_{\overline{\mathbb{F}}_p}$.
4. At a supersingular point x of $(X_p)_{\overline{\mathbb{F}}_p}$, the completion of the local ring is isomorphic to

$$W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p^{k_x}),$$

where k_x is the order of the automorphism group of the false elliptic curve with level structure corresponding to the point x modulo $\{\pm 1\}$ if -1 is in Γ , and the order of the full automorphism group if -1 is not in Γ .

Proof: Statement 2 is well-known, and statement 1 is known except for the regularity of X_p at nonsupersingular points on the characteristic p fiber. It thus suffices to prove this regularity, along with statements 3 and 4. The arguments we give are basically adaptations of arguments found in [DR] for the case when $D = 1$. Let $r > 3$ be a prime that does not divide NDp , and consider the curves $X_r = X^D(\Gamma \cap \Gamma_1(r))$ and $X_{p,r} = X^D(\Gamma_0(p) \cap \Gamma \cap \Gamma_1(r))$. Then X_r and $X_{p,r}$ are fine moduli spaces, so the above theorem applies.

We have maps $X_r \rightarrow X$ and $X_{p,r} \rightarrow X_p$ obtained by forgetting the level structure at r . These maps are finite, and ramified precisely at those x in X or X_p for which $k_x > 1$. Moreover, at any x in X (resp. X_p) for which $k_x > 1$,

the automorphism group of x acts on the completion of the local ring of any preimage of x in X_r (resp. $X_{p,r}$). This action is faithful if -1 is not in Γ , and has kernel $\{\pm 1\}$ if -1 is in Γ . Let G_x denote the group $\text{Aut}(x)$ in the former case, and $\text{Aut}(x)/\{\pm 1\}$ in the latter. Then the above maps induce isomorphisms:

$$\begin{aligned}\hat{\mathcal{O}}_{X,x} &\cong \hat{\mathcal{O}}_{X_r,\tilde{x}}^{G_x} \\ \hat{\mathcal{O}}_{X_{p,r},x} &\cong \hat{\mathcal{O}}_{X_{p,r},\tilde{x}}^{G_x},\end{aligned}$$

where \tilde{x} is any lift of x to X_r or $X_{p,r}$, respectively, and $\hat{\mathcal{O}}_{X,x}$ is the completion of the strict Henselization of the local ring of X at x .

Suppose x is a point on the characteristic p fiber of X_p . If \tilde{x} is a regular point of $X_{p,r}$, then $\hat{\mathcal{O}}_{X_{p,r},\tilde{x}}$ is isomorphic to $W(\overline{\mathbb{F}}_p)[[X]]$. By the same argument as the proof of [DR], VI.6.9, the subring of G_x -invariants of this ring is again a power series ring in one variable. In particular, x is a regular point. This proves statement 1.

If \tilde{x} is a supersingular point of $X_{p,r}$, then $\hat{\mathcal{O}}_{X_{p,r},\tilde{x}}$ is isomorphic to

$$W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p).$$

Again, the same argument as the proof of [DR], VI.6.9 shows that the subring of G_x -invariants is isomorphic to $W(\overline{\mathbb{F}}_p)[[X, Y]]/(XY - p^{k_x})$. This proves statement 4.

For statement 3, we have a commutative diagram

$$\begin{array}{ccc}(X_r)_{\overline{\mathbb{F}}_p} & \longrightarrow & (X_{p,r})_{\overline{\mathbb{F}}_p} \\ \downarrow & & \downarrow \\ X_{\overline{\mathbb{F}}_p} & \longrightarrow & (X_p)_{\overline{\mathbb{F}}_p}\end{array}$$

in which the horizontal maps can be either both Frob or both Ver. Since the map from $X_{p,r}$ to X_p is a surjection, X_p will thus be the union of the images of Frob and Ver, and these images will intersect precisely at the supersingular points of X_p .

All that remains is to prove that Frob and Ver are closed immersions. Let π_1 and π_2 be the two degeneracy maps $(\mathcal{X}_p)_{\overline{\mathbb{F}}_p} \rightarrow \mathcal{X}_{\overline{\mathbb{F}}_p}$. Then π_1 Frob and π_2 Ver are automorphisms of $\mathcal{X}_{\overline{\mathbb{F}}_p}$. Thus the same is true on the underlying coarse moduli spaces, so Frob and Ver are closed immersions. ■

References

- [Bu1] K. Buzzard, *Integral models of certain Shimura curves*, Duke Mathematical Journal **87** (1997), 591–612.
- [Ca] H. Carayol, *Formes modulaires et représentations Galoisienne à valeurs dans un anneau local complet*, in *p-Adic Monodromy and the Birch and Swinnerton-Dyer Conjecture* (Boston, MA, 1991), (B. Mazur and G. Stevens, eds.), Contemporary Mathematics, 165, American Mathematical Society, Providence, RI, 1991, pp. 213–238.
- [CS] B. Conrad and W. Stein, *Component groups of purely toric quotients*, Mathematical Research Letters **8** (2001), 745–766.
- [DR] P. Deligne and M. Rapoport, *Schémas de modules des courbes elliptiques*, in *Lecture Notes in Mathematics* Vol. 349, 1971, pp. 123–165.
- [Di] F. Diamond, *The refined conjecture of Serre*, in *Elliptic Curves, Modular Forms, and Fermat's Last Theorem*, (J. Coates and S. T. Yau, eds.), International Press, 1995, pp. 22–37.
- [DT] F. Diamond and R. Taylor, *Nonoptimal levels of mod l modular representations*, Inventiones Mathematicae **115** (1994), 435–462.
- [Em] M. Emerton, *Optimal quotients of modular Jacobians*, Mathematische Annalen **327** (2003), 429–458.
- [Fa] G. Faltings, *Finiteness theorems for abelian varieties*, in *Arithmetic Geometry*, (G. Cornell and J. Silverman, eds.), Springer-Verlag, New York, 1986, pp. 9–27.
- [SGA] A. Grothendieck, *SGA 7 I, Exposé IX*, in *Lecture Notes in Mathematics* Vol. 288, Springer, Berlin–New York, 1972, pp. 313–523.
- [J-L] H. Jacquet and R. P. Langlands, *Automorphic forms on $GL(2)$* , in *Lecture Notes in Mathematics* Vol. 114, Springer, Berlin–New York, 1970.
- [Ki] L. Kilford, *Some non-Gorenstein Hecke algebras attached to spaces of modular forms*, Journal of Number Theory **97** (2002), 157–164.
- [Ko] D. Kohel, *Hecke module structure of quaternions*, in *Class Field Theory, Centenary and Prospect*, (K. Miyake, ed.), Advanced Studies in Pure Mathematics 30, Mathematical Society of Japan, Tokyo, 2001, pp. 177–195.
- [Ma] B. Mazur, *Modular curves and the Eisenstein ideal*, Institut des Hautes Études Scientifiques, Publications Mathématiques **47** (1977), 33–186.
- [Ri3] K. Ribet, *Sur les variétés abéliennes à multiplications réelles*, Comptes Rendus de l'Académie des Sciences, Paris Série A–B **291** (1980), A121–A123.
- [Ri2] K. Ribet, *Multiplicities of Galois representations in Jacobians of Shimura curves*, Israel Mathematical Conference Proceedings Vol. 3, Weizmann, Jerusalem, 1990, pp. 221–236.

- [Ri1] K. Ribet, *On modular representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms*, *Inventiones Mathematicae* **100** (1990), 431–476.
- [Ri4] K. Ribet, *Irreducible Galois representations arising from component groups of Jacobians*, in *Elliptic Curves, Modular Forms, and Fermat's Last Theorem* (Hong Kong, 1993), Ser. Number Theory I, International Press, Cambridge, Ma, 1995, pp. 131–147.
- [Sh] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, Princeton, N.J., 1971.
- [Ti] J. Tilouine, *Hecke algebras and the Gorenstein property*, in *Modular Forms and Fermat's Last Theorem*, Springer, New York, 1997, pp. 327–342.
- [Wi] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, *Annals of Mathematics* 2 **131** (1990), 443–551.
- [Ya] L. Yang, *Multiplicities of Galois representations in the higher weight sheaf cohomology associated to Shimura curves*, Ph.D. thesis, City University of New York, New York, 1996.